

On the spatial asymptotic decay of a suitable weak solution to the Navier-Stokes Cauchy problem

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Abstract - We prove space-time decay estimates of suitable weak solutions to the Navier-Stokes Cauchy problem, corresponding to a given asymptotic behavior of the initial data of the same order of decay. We use two main tools. The first is a result obtained in [4] on the behavior of the solution in a neighborhood of $t = 0$ in the L_{loc}^∞ -norm, which enables us to furnish a representation formula for a suitable weak solution. The second is the asymptotic behavior of $\|u(t)\|_{L^2(\mathbb{R}^3 \setminus B_R)}$ for $R \rightarrow \infty$. Following a Leray's point of view, roughly speaking our result proves that a possible space-time turbulence does not perturb the asymptotic spatial behavior of the initial data of a suitable weak solution.

Keywords: Navier-Stokes equations, suitable weak solutions, space-time asymptotic behavior.

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1 Introduction

In this paper, we study the initial value problem:

$$\begin{aligned} v_t + v \cdot \nabla v + \nabla \pi_v &= \Delta v, \quad \nabla \cdot v = 0, \quad \text{in } (0, T) \times \mathbb{R}^3, \\ v(0, x) &= v_o(x) \quad \text{on } \{0\} \times \mathbb{R}^3. \end{aligned} \quad (1.1)$$

In system (1.1) v is the kinetic field, π_v is the pressure field, $v_t := \frac{\partial}{\partial t} v$ and $v \cdot \nabla v := v_k \frac{\partial}{\partial x_k} v$. For brevity, we assume zero body force. We set $J^q(\mathbb{R}^3) :=$ completion of $\mathcal{C}_0(\mathbb{R}^3)$ with respect to the L^q -norm, $q \in (1, \infty)$. The symbol $\mathcal{C}_0(\mathbb{R}^3)$ denotes the subset of $C_0^\infty(\mathbb{R}^3)$ whose elements are divergence free. By P_q (the index q is omitted when there is no danger of confusion) we mean the projector from L^q into J^q . For properties and details on these spaces see for instance [6]. Moreover, we set $J^{1,q}(\mathbb{R}^3) :=$ completion of $\mathcal{C}_0(\mathbb{R}^3)$ with respect to the $W^{1,q}$ -norm. For a nonnegative integer m , if X is a Banach space, the symbols $C^m(a, b; X)$ and $L^p(a, b; X)$ mean the spaces of functions defined in $(a, b) \subseteq \mathbb{R}$ with value in the Banach space X , that are m -times continuous differentiable in $[a, b]$ and L^p -integrable on (a, b) , respectively.

We use the same symbol to denote vector or scalar functions and function spaces. We set $(u, g) := \int_{\mathbb{R}^3} u \cdot g dx$.

Definition 1.1 A pair (v, π_v) , such that $v : (0, T) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $\pi_v : (0, T) \times \mathbb{R}^3 \rightarrow \mathbb{R}$, is said a weak solution to problem (1.1) if

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i) for all $T > 0$, $v \in L^2(0, T; J^{1,2}(\mathbb{R}^3))$ and $\pi_v \in L^{\frac{5}{3}}((0, T) \times \mathbb{R}^3)$,

$$\|v(t)\|_2^2 + 2 \int_s^t \|\nabla v(\tau)\|_2^2 d\tau \leq \|v(s)\|_2^2, \quad \forall t \geq s, \text{ for } s = 0 \text{ and a.e. in } s \geq 0,$$

ii) $\lim_{t \rightarrow 0} \|v(t) - v_0\|_2 = 0$,

iii) for all $t, s \in (0, T)$, the pair (v, π_v) satisfies the equation:

$$\int_s^t \left[(v, \varphi_\tau) - (\nabla v, \nabla \varphi) + (v \cdot \nabla \varphi, v) + (\pi_v, \nabla \cdot \varphi) \right] d\tau + (v(s), \varphi(s)) = (v(t), \varphi(t)),$$

for all $\varphi \in C_0^1([0, T] \times \mathbb{R}^3)$.

As it is known, the regularity and uniqueness of a weak solution are still open problems. However, in order to improve the results of regularity, in the fundamental paper [1], Caffarelli, Kohn and Nirenberg introduce the notion of suitable weak solution:

Definition 1.2 A pair (v, π_v) is said a suitable weak solution if it is a weak solution in the sense of the Definition 1.1 and, moreover,

$$\begin{aligned} \int_{\mathbb{R}^3} |v(t)|^2 \phi(t) dx + 2 \int_s^t \int_{\mathbb{R}^3} |\nabla v|^2 \phi dx d\tau &\leq \int_{\mathbb{R}^3} |v(s)|^2 \phi(s) dx \\ &+ \int_s^t \int_{\mathbb{R}^3} |v|^2 (\phi_\tau + \Delta \phi) dx d\tau + \int_s^t \int_{\mathbb{R}^3} (|v|^2 + 2\pi_v) v \cdot \nabla \phi dx d\tau, \end{aligned} \quad (1.2)$$

for all $t \geq s$, for $s = 0$ and a.e. in $s \geq 0$, and for all nonnegative $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$.

In [1] and [13] the following existence result is proved:

Theorem 1.1 For all $v_0 \in J^2(\Omega)$ there exists a suitable weak solution.

Further in the same paper [1] it is proved that in the set of suitable weak solutions a partial regularity result holds. Among others things, in [1] the authors prove that under the further assumption $\nabla v_0 \in L^2(\mathbb{R}^3 \setminus B_{R_0})$ a suitable weak solution is regular in a neighborhood of “infinity”, that is for $|x| > M_0 R_0$, $M_0 > 1$. Roughly speaking, this result is the analogous of the one related to the *structure theorem* by Leray, with the difference that Leray’s result is on regularity of a weak solution in neighborhood of $t = +\infty$. Still in the light of the analogy, roughly speaking, following Leray, we say that the possible *turbulence* of a weak solution not only appears in a finite time but also in a bounded region of the space, whose parabolic one-dimensional Hausdorff measure is null (this last is true as soon as a suitable weak solution exists). Actually, the smallness of the data for large $|x|$, although given by means of integrability conditions outside a ball, preserves the regularity of the weak solution (as for “small data”).

In the wake of the previous results, in [4] we have proved a result concerning the behavior in time of the L_{loc}^∞ norm of the solution in a neighborhood of $t = 0$ for suitable weak solutions, corresponding to a suitably small data (see Theorem 2.1 below).

The aim of this note goes in the direction of the last claims. We prove that, not only the possible *turbulence* does not perturb the regularity of a weak solution in a neighborhood of $t = +\infty$, but, if an asymptotic spatial behavior of the initial data v_o is given, then the same behavior holds for a suitable weak solution for all $t > 0$. As far as we know, such a property to date was ensured only for small data (cf. [3], [7]).

The theorem we are going to state is the main result of the paper:

Theorem 1.2 *Let $v_o \in J^2(\mathbb{R}^3)$ and, for some $\alpha \in [1, 3)$ and $R_0 > 0$, let be $|v_o(x)| \leq V_o|x|^{-\alpha}$, for $|x| > R_0$. Let (v, π_v) be a suitable weak solution to the Navier-Stokes Cauchy problem. Then, there exists a constant $M \geq 1$ such that*

$$|v(t, x)| \leq c(v_o)|x|^{-\alpha}, \text{ for all } (t, x) \in (0, \infty) \times \mathbb{R}^3 \setminus B_{MR_0}, \quad (1.3)$$

where M is independent of v_o and $c(v_o)$ depends on V_o and $\|v_o\|_2$.

Corollary 1.1 *For a solution of Theorem 1.2, for all $\beta \in [0, \alpha]$, we get:*

$$|v(t, x)| \leq c(v_o)|x|^{-\alpha+\beta}t^{-\frac{\beta}{2}}, \text{ for all } (t, x) \in (0, \infty) \times \mathbb{R}^3 \setminus B_{MR_0}; \quad (1.4)$$

and

$$|v(t, x)| \leq c(v_o)|x|^{-\alpha+\beta}t^{-\frac{\beta}{2}}, \text{ for all } (t, x) \in (T_0, \infty) \times \mathbb{R}^3, \quad (1.5)$$

where $T_0 \leq c\|v_o\|_2^4$.

Here we state Theorem 1.2 for solutions to the Navier-Stokes Cauchy problem just for the sake of brevity. The result of Theorem 1.2 can be seen as a continuous dependence of the null solution. In this sense the theorem is a continuation of the one proved in [4], that we employ here for regularity, see Theorem 2.1 below.

In a forthcoming paper, the same result will be proved for a three dimensional exterior domain Ω and for weak solutions with initial data in $J^3(\Omega)$, hence not necessarily with finite energy! This assumption seems to be more coherent with the assumption $|v_o(x)| \leq V_o|x|^{-\alpha}$, $\alpha \in [1, 3)$.

As far as we know, the technique employed to prove Theorem 1.2 is original in the framework of the ones employed to prove a spatial asymptotic behavior of a solution $v(t, x)$, more in general, to a parabolic equation. Indeed, it essentially follows from two properties. The former, well known, concerns the spatial behavior of the solution $w(t, x)$ to the heat equation. The latter is connected with the asymptotic behavior of the functional:

$$\int_{|y|>c|x|} |u(t, y)|^2 dy \leq c(t)|x|^{-1}, \quad (1.6)$$

where $u = v - w$, which we think to be an original tool for the spatial behavior. Estimate (1.6) comes from estimate (1.2) for a suitable $\phi(x)$, written for the difference u . For the special ϕ , we like to call the above functional as Leray's generalized energy inequality. It was used by Leray in [9] for a compactness property.

We conclude the introduction by quoting paper [5], where a similar result is given. More precisely, for the Navier-Stokes initial boundary value problem in an exterior domain $\Omega \subseteq \mathbb{R}^3$ it is proved:

Assume that the initial data $v_0 \in J^2(\Omega) \cap J^{\frac{5}{4}}(\Omega)$ with $|v_0(x)| \leq V_0|x|^{-\alpha}$, for some $\alpha \in [\frac{7}{6}, 3)$ and $|x| > R_0 > 0$. Assume that, $\bar{\gamma} \in (0, \frac{1}{4}]$, $v_0 \in D(A_2^{\frac{3}{4}+\bar{\gamma}}) \cap J^{\frac{9}{8}}(\Omega)$. Then, there are suitable constants c_0 and \bar{R}_0 such that

$$|v(t, x)| \leq c_0|x|^{-\min\{\alpha, \frac{13}{5}\}}, \text{ for all } (t, x) \in (0, \infty) \times \mathbb{R}^3 \setminus B_{\bar{R}_0}.$$

Here, $A_2 := P_2\Delta$ is the Stokes operator, and $D(A_2^s)$ is the domain of definition of the fractional power s of A_2 .

We point out that key ideas, the technique and the proofs in [5] and in the present paper are completely different.

The plan of the paper is the following. In order to perform pointwise estimates, in sec. 2 we establish results of partial regularity based on the results of [4]. In sec. 3 we recall classical results concerning the solutions to the Stokes Cauchy problem. In sec. 4 we prove estimate (1.6) which is strategic for our aims. In sec. 5 we give the representation formula of the solutions to the Navier-Stokes Cauchy problem that we employ in sec. 6 and sec. 7 to prove our results.

2 Preliminary results on partial regularity of a suitable weak solution

Throughout the paper, where it is appropriate, we give an explicit dependence of the constants from the L^2 -norm of the data. In the other cases, the dependence will be referred to simply by $c(v_o)$.

Lemma 2.1 *In the hypotheses of Theorem 1.2*

$$\text{for all } \varepsilon > 0, \int_{\mathbb{R}^3} \frac{|v_o(y)|^2}{|x-y|} dy \leq \frac{c}{\varepsilon|x|} \|v_o\|_2^2 + \frac{c}{|x|^{\frac{1}{2}}} V_o \|v_o\|_2, \quad |x| > \frac{R_0}{1-\varepsilon}, \quad (2.1)$$

with c independent of x and v_o .

Proof. We start by proving (2.1) for $\alpha = 1$. Then, *a fortiori*, it holds for $\alpha \in (1, 3)$. Given $\varepsilon > 0$ and $x \in \mathbb{R}^3$, by virtue of our assumption, we easily deduce that

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|v_o|^2}{|x-y|} dy &\leq \int_{|y| < (1-\varepsilon)|x|} \frac{|v_o|^2}{|x-y|} dy + cV_o \int_{(1-\varepsilon)|x| < |y|} \frac{|v_o|}{|x-y|(1+|y|)} dy \\ &\leq \frac{\|v_o\|_2^2}{\varepsilon|x|} + c \left[\int_{(1-\varepsilon)|x| < |y|} \frac{(1+|y|)^{-2}}{|x-y|^2} dy \right]^{\frac{1}{2}} V_o \|v_o\|_2, \end{aligned} \quad (2.2)$$

which implies the thesis. □

Let $x_0 \in \mathbb{R}^3$ and $R_0 > 0$. Let $v_o \in J^2(\mathbb{R}^3)$. We set

$$\mathcal{E}_0(x_0, R_0) := \operatorname{ess\,sup}_{B(x_0, R_0)} \|v_o\|_{w(x)} := \left\| \left(\int_{\mathbb{R}^3} \frac{|v_o(y)|^2}{|x-y|} dy \right)^{\frac{1}{2}} \right\|_{L^\infty(B(x_0, R_0))}.$$

Theorem 2.1 *Let (v, π_v) be a suitable weak solution corresponding to $v_o \in J^2(\mathbb{R}^3)$. There exist absolute constants ε_1 , C_1 and C_2 such that, if*

$$C_1 \mathcal{E}_0(x_0, R_0) < 1 \text{ and } C_2(\mathcal{E}_0^3 + \mathcal{E}_0^{\frac{5}{2}}) \leq \varepsilon_1, \quad (2.3)$$

then

$$|v(t, x)| \leq c(\mathcal{E}_0^3 + \mathcal{E}_0^{\frac{5}{2}})^{\frac{1}{3}} t^{-\frac{1}{2}}, \quad (2.4)$$

provided that (t, x) is a Lebesgue point with $\|v_o\|_{w(x)} < \infty$ and $x \in B(x_0, R_0)$.

Proof. See [4], Theorem 1.1. \square

We complete the results concerning a suitable weak solution with the following lemma on the regularity and the asymptotic behavior of the solutions:

Lemma 2.2 *If $v_o \in J^2(\mathbb{R}^3) \cap J^p(\mathbb{R}^3)$, $p \in (1, 2]$, then there exists a $T_0 \leq c\|v_o\|_2^4$ such that*

$$\|v(t)\|_2 = \begin{cases} o(1) & \text{if } p = 2, \\ c(v_o)t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}, t > 0 & \text{if } p \in (1, 2). \end{cases} \quad (2.5)$$

$$\|v(t)\|_\infty \leq c\|v_o\|_2 t^{-\frac{3}{4}}, t > T_0.$$

Proof. Estimates (2.5)_{1,2} can be found in [10], instead (2.5)₃ is well known (see Leray [9]).

Lemma 2.3 *In the hypotheses of Theorem 1.2 there exists a constant $M_0 \geq 1$ such that*

$$|v(t, x)| \leq c(\mathcal{E}_0^3 + \mathcal{E}_0^{\frac{5}{2}})^{\frac{1}{3}} t^{-\frac{1}{2}}, \text{ a.e. in } t > 0 \text{ and } |x| > M_0 R_0, \quad (2.6)$$

provided that (t, x) is a Lebesgue point.

Proof. It is enough to verify that the hypotheses of Theorem 2.1 are satisfied. To this end, we employ Lemma 2.1 which ensures the existence of $\overline{R}(C_1, C_2, \varepsilon_1)$ such that, for $|x| > \overline{R}$, $[\int_{\mathbb{R}^3} \frac{|v_o(x)|^2}{|x-y|} dy]^{\frac{1}{2}}$ satisfies (2.3). Setting $\overline{R} := M_0 R_0$, we have proved the lemma. \square

As a consequence of the above lemmas on the L^∞ -norm of a suitable weak solution we can claim

Corollary 2.1 *In the hypotheses of Theorem 1.2, we get*

$$\|v(t)\|_{L^\infty(|x| > M_0 R_0)} \leq c(v_o, T_0) t^{-\frac{3}{4}}, t > 0. \quad (2.7)$$

Lemma 2.4 *If (v, π_v) is a suitable weak solution, then, the pressure field admits the representation formula:*

$$\pi_v(t, x) = -D_{x_i} D_{x_j} \int_{\mathbb{R}^3} \mathcal{E}(x - y) v^i(y) v^j(y) dy =: \mathbb{E}[v, v](t, x), \quad (2.8)$$

a.e. in $(t, x) \in (0, \infty) \times \mathbb{R}^3$.

Proof. See [4], Lemma 4.1.

3 Some lemmas on the Stokes Cauchy problem

We consider the Stokes Cauchy problem

$$w_t - \Delta w = -\nabla \pi_w, \quad \nabla \cdot w = 0 \text{ in } (0, T) \times \mathbb{R}^3, \quad w(0, x) = v_o(x) \text{ on } \{0\} \times \mathbb{R}^3. \quad (3.1)$$

We denote by

$$H_{ij}(s, z) := \delta_{ij} H(s, z) := \delta_{ij} (4\pi s)^{-\frac{3}{2}} e^{-\frac{|z|^2}{4s}} \quad (3.2)$$

the general component of the heat kernel tensor, and set

$$\mathbb{H}[w(s)](t-s, x) := \int_{\mathbb{R}^3} H(t-s, x-y) w(s, y) dy.$$

Then, for the solution of (3.1) we have $w(t, x) := \mathbb{H}[v_o](t, x)$. Moreover, we recall the estimate (k nonnegative integer and β multi-index):

$$|D_s^k D_z^\beta H(s, z)| \leq c(|z| + s^{\frac{1}{2}})^{-3-2k-|\beta|}. \quad (3.3)$$

We are interested in the following result:

Lemma 3.1 *In the hypotheses of Theorem 1.2 on v_o ,*

for all $T > 0, w \in C(0, T; L^2(\mathbb{R}^3))$,

for all $\eta > 0, k \geq 0, |\beta| \geq 0, D_t^k D_x^\beta w \in C(\eta, T; C_b(\mathbb{R}^3)) \cap C(\eta, T; L^2(\mathbb{R}^3))$,

$$\|w(t)\| + 2 \int_s^t \|\nabla w(\tau)\|^2 d\tau = \|w(s)\|^2, \text{ for all } t \geq s \geq 0. \quad (3.4)$$

Moreover, there holds

$$|w(t, x)| \leq c \min\{\|v_o\|_2, V_o\} \min\left\{\frac{1}{(1+t)^{\frac{\alpha}{2}}}, \frac{1}{(1+|x|)^\alpha}\right\}, \quad t > 0, |x| > \max\{2R_0, 1\}. \quad (3.5)$$

Proof. Properties (3.4) are well known. To prove (3.5), we employ the representation formula and (3.3), so that

$$\begin{aligned} |w(t, x)| &\leq \int_{\mathbb{R}^3} H(t, x-y) |v_o(y)| dy = \int_{B(R_0)} H(t, x-y) |v_o(y)| dy + V_o \int_{\mathbb{R}^3 \setminus B(R_0)} H(t, x-y) (1+|y|)^{-\alpha} dy \\ &\leq c(R_0) \|v_o\|_2 (|x| + t^{\frac{1}{2}})^{-3} + cV_o \min\{(1+t)^{-\frac{\alpha}{2}}, (1+|x|)^{-\alpha}\}, \end{aligned}$$

provided that $|x| > \max\{2R_0, 1\}$, which proves the lemma. \square

We also approach problem (3.1) in a form weaker than the usual one for the initial value problem for the Stokes equations. This weak formulation, introduced in [11], allows to consider initial data in the Lebesgue spaces L^p , $p \in [1, \infty]$, and not in the space of the hydrodynamics J^p , $p \in (1, \infty)$. Its interest is connected with the possibility of deducing estimates in L^r -spaces with $r \in (1, \infty]$ by means of duality arguments. Of course, for an initial data in J^p we come back to the classical Stokes solutions.

We have the following special result, for a general formulation see [11] (such a solution is denoted by (θ, π_θ)):

Lemma 3.2 Let $\theta_0 \in C_0(\mathbb{R}^3)$. Then, to the data θ_0 there corresponds a unique smooth solution (θ, c_0) to the Cauchy problem (3.1) such that $\theta \in \bigcap_{q>1} C(0, T; J^q(\mathbb{R}^3))$, $\theta \in \bigcap_{q>1} L^q(\eta, T; W^{2,q}(\mathbb{R}^3))$ and $\theta_t \in \bigcap_{q>1} L^q(\eta, T; L^q(\mathbb{R}^3))$, $\eta > 0$. Moreover, for $q \in (1, \infty]$,

$$\begin{aligned} \|\theta(t)\|_q &\leq c\|\theta_0\|_1 t^{-\mu}, & \mu &= \frac{3}{2}\left(1 - \frac{1}{q}\right), \quad t > 0; \\ \|\nabla\theta(t)\|_q &\leq c\|\theta_0\|_1 t^{-\mu_1}, & \mu_1 &= \frac{1}{2} + \mu \\ \|\theta_t(t)\|_q &\leq c\|\theta_0\|_1 t^{-\mu_2}, & \mu_2 &= 1 + \mu, \quad t > 0; \end{aligned} \quad (3.6)$$

with c independent of θ_0 . Finally, $\lim_{t \rightarrow 0} (\theta(t) - \theta_0, \varphi) = 0$ holds for any $\varphi \in C^1(\mathbb{R}^3) \cap J^{1,q'}(\mathbb{R}^3)$.

Proof. See [11], Lemma 3.2. □

Corollary 3.1 In the hypotheses of Lemma 3.2, for all $\lambda \in (0, 1]$, the following estimates hold:

$$\begin{aligned} \|\theta(t) - \theta(s)\|_q &\leq c\xi^{-\lambda - \frac{3}{2}(1 - \frac{1}{q})} |t - s|^\lambda \|\theta_0\|_1, \\ \|\nabla\theta(t) - \nabla\theta(s)\|_q &\leq c\xi^{-\frac{3}{2}\lambda - \frac{3}{2}(1 - \frac{1}{q})} |t - s|^\lambda \|\theta_0\|_1, \end{aligned} \quad (3.7)$$

$t, s > 0$, where $\xi := \min\{s, t\}$.

Proof. Let $\xi := \min\{s, t\}$. From (3.6)₁, $\|\theta(\frac{\xi}{2})\|_q \leq c\|\theta_0\|_1 \xi^{-\mu}$. On the other hand, from the representation formula, one has $\theta(t, x) = \mathbb{H}[\theta(\frac{\xi}{2})](t - \frac{\xi}{2}, x)$. Hence, using the L^q -Hölder's properties of θ , for all $\lambda \in (0, 1]$, we get

$$\|\theta(t) - \theta(s)\|_q \leq c\xi^{-\lambda} |t - s|^\lambda \|\theta(\frac{\xi}{2})\|_q \leq c\xi^{-\lambda - \frac{3}{2}(1 - \frac{1}{q})} |t - s|^\lambda \|\theta_0\|_1.$$

Similar arguments lead to estimate (3.7)₂. □

4 A space time behavior of the Leray's generalized energy inequality

We start by proving the following interpolation inequality, of the same kind of the one by Gagliardo and Nirenberg. It is a particular case of a more general result for exterior domains, obtained in [2]. The difference with respect to the usual result is that the function u does not belong to a completion space of $C_0^\infty(\mathbb{R}^3)$.

Lemma 4.1 Let $u \in W^{1,2}(\mathbb{R}^3 \setminus B_R)$. Then there exists a constant c independent of u and R such that, for any $p \in [2, 6]$,

$$\|u\|_{L^p(|x| \geq R)} \leq c \|\nabla u\|_{L^2(|x| \geq R)}^a \|u\|_{L^2(|x| \geq R)}^{1-a}, \quad a = \frac{3(p-2)}{2p}. \quad (4.1)$$

Proof. Let $x \in \mathbb{R}^3 \setminus B_R$ be the vertex of an infinite cone $C_x \subset \mathbb{R}^3 \setminus B_R$, of fixed aperture independent of x . Let (r, θ) be spherical polar coordinates with origin at x , assume that the cone C_x is given by $r \in (0, \infty)$ and $\theta \in \Theta$, and let $r^2 \omega(\theta) dr d\theta$ the volume

element. Let $\{h_\rho(r)\}$ be a sequence of smooth cut-off functions such that $h_\rho(r) \in [0, 1]$, $h_\rho(r) = 1$ for $r \leq \rho$, $h_\rho(r) = 0$ for $r \geq 2\rho$, and $|h'_\rho(r)| \leq c\rho^{-1}$. Then

$$|u(x)| = |u(0, \theta)| = \left| - \int_0^\infty \frac{\partial}{\partial r}(u(r, \theta)h(r))dr \right| \leq \int_0^\infty |\nabla u(r, \theta)h(r)|dr + \frac{c}{\rho} \int_0^\infty |u(r, \theta)|dr.$$

Multiplying by $\omega(\theta)$ and integrating over Θ we get

$$4\pi|u(x)| \leq \int_{C_x} \frac{|\nabla u(y)|}{|x-y|^2} dy + \frac{c}{\rho} \int_{C_x} \frac{|u(y)|}{|x-y|^2} dy \leq \int_{|y| \geq R} \frac{|\nabla u(y)|}{|x-y|^2} dy + \frac{c}{\rho} \int_{|y| \geq R} \frac{|u(y)|}{|x-y|^2} dy.$$

We let ρ tend to infinity and then apply the Hardy-Littlewood-Sobolev theorem, and we find

$$\|u\|_{L^6(|x| \geq R)} \leq c \|\nabla u\|_{L^2(|x| \geq R)}, \quad (4.2)$$

with c independent of R . Using the interpolation between Lebesgue spaces

$$\|u\|_{L^p(|x| \geq R)} \leq c \|u\|_{L^6(|x| \geq R)}^a \|u\|_{L^2(|x| \geq R)}^{1-a}, \quad a = \frac{3(p-2)}{2p},$$

and then estimate (4.2), we arrive at (4.1). \square

We set

$$u := v - w \quad \text{and} \quad \pi_u := \pi_v, \quad (4.3)$$

where (v, π_v) is a suitable weak solution to the Navier-Stokes Cauchy problem and w is the solution to the Stokes Cauchy problem. Both the solutions assume the initial data v_\circ . We define the symbol $\|\cdot\|_{L^p(k, R)} := \|\cdot\|_{L^p(|y| > \frac{k}{k+1}R)}$, for all $k \geq 0$.

Lemma 4.2 *In the hypotheses of Theorem 1.2, for all $k \geq 2$ and for $R > 4R_0$, there exists a $c(k)$ such that*

$$\|\pi_v\|_{L^2(k, R)} \leq c \|u\|_{L^2(k-1, R)}^{\frac{1}{2}} \|\nabla u\|_{L^2(k-1, R)}^{\frac{3}{2}} + \frac{V_\circ}{R^\alpha} \|u\|_{L^2(k-1, R)} + c \frac{V_\circ^2}{R^{2\alpha - \frac{3}{2}}} + \frac{1}{R^{\frac{3}{2}}} \|v_\circ\|_2^2, \quad (4.4)$$

almost everywhere in $t > 0$.

Proof. From (4.3) and the representation formula (2.8) of the pressure field we get

$$\pi_v = \mathbb{E}(v \otimes v) = \mathbb{E}(u \otimes u) + \mathbb{E}(u \otimes w) + \mathbb{E}(w \otimes u) + \mathbb{E}(w \otimes w) = \sum_{i=1}^4 \pi^i. \quad (4.5)$$

For $i = 1, \dots, 4$ and $|x| > \frac{k}{k+1}R$, we get, with obvious meaning of the symbols, the estimate

$$|\pi^i(x)| \leq c \int_{|y| < \frac{k-1}{k}R} \frac{|a||b|}{|x-y|^3} dy + \left| D_{x_h x_k} \int_{|y| > \frac{k-1}{k}R} \mathcal{E}(x-y) a_h b_k dy \right| =: \pi_1^i + \pi_2^i.$$

By the assumption on x there holds

$$|x-y| \geq |x| - |y| \geq \frac{|x|}{k^2}, \quad \text{for all } |y| < \frac{k-1}{k}R,$$

and we get

$$|\pi_1^i(x)| \leq \frac{c(k)}{|x|^3} \int_{|y| < \frac{k-1}{k}R} |u||w|dy \leq \frac{c(k)}{|x|^3} \|u\|_2 \|w\|_2, \quad i = 2, 3,$$

which implies

$$\|\pi_1^i\|_{L^2(k,R)} \leq \frac{c(k)}{R^{\frac{3}{2}}} \|u\|_2 \|v_\circ\|_2, \quad i = 2, 3.$$

For the term π_2^i , by applying the Calderón-Zigmund theorem and then Lemma 3.1, we get

$$\|\pi_2^i\|_{L^2(k,R)} \leq c \|u\| \|w\|_{L^2(k-1,R)} \leq c \frac{V_\circ}{R^\alpha} \|u\|_{L^2(k-1,R)}, \quad i = 2, 3.$$

Repeating the above arguments for the term π^4 , we get

$$|\pi_1^4(x)| \leq \frac{c(k)}{|x|^3} \|v_\circ\|_2^2,$$

hence

$$\|\pi_1^4\|_{L^2(k,R)} \leq \frac{c(k)}{R^{\frac{3}{2}}} \|v_\circ\|_2^2.$$

Since $\alpha \geq 1$ and $R > 4R_0$, from Lemma 3.2 we easily deduce

$$\|\pi_2^4\|_{L^2(k,R)} \leq c \|w^2\|_{L^2(k-1,R)} \leq c V_\circ^2 R^{-2\alpha + \frac{3}{2}}.$$

Finally, we estimate π^1 . For π_1^1 we obtain the same estimate, that is

$$|\pi_1^1(x)| \leq \frac{c(k)}{|x|^3} \|u\|_2^2 \leq \frac{c(k)}{|x|^3} \|v_\circ\|_2^2.$$

Hence, we get

$$\|\pi_1^1\|_{L^2(k,R)} \leq \frac{c(k)}{R^{\frac{3}{2}}} \|v_\circ\|_2^2.$$

Applying the Calderón-Zigmund theorem for singular integrals and estimate (4.1), we get

$$\|\pi_2^1\|_{L^2(k,R)} \leq c \|u\|_{L^4(k-1,R)}^2 \leq c \|u\|_{L^2(k-1,R)}^{\frac{1}{2}} \|\nabla u\|_{L^2(k-1,R)}^{\frac{3}{2}}.$$

The above estimate and formula (4.5) give (4.4). \square

Lemma 4.3 *In the hypotheses of Theorem 1.2, for all $k \geq 2$ and for $R > 2M_0R_0$, there exists a constant c such that*

$$\|\pi_v\|_{L^2(k,R)} \leq c (\|v\|_2^2 + \|v\|_{L^4(k-1,R)}^2), \quad (4.6)$$

almost everywhere in $t > 0$.

Proof. Estimate (4.6) is obtained by the same arguments used in the proof of Lemma 4.2, provided that we consider $\mathbb{E}[v, v]$ and not its decomposition by means of u, w . The constant c depends on k and R , however it is bounded with respect either to k and R . \square

Lemma 4.4 *Assume that (v, π_v) is a suitable weak solution. Then it satisfies the following inequality*

$$\begin{aligned} \int_{\mathbb{R}^3} |v(t)|^2 \psi(t) dx + 2 \int_s^t \int_{\mathbb{R}^3} |\nabla v(\tau)|^2 \psi dx d\tau &\leq \int_{\mathbb{R}^3} |v(s)|^2 \psi(s) dx \\ &+ \int_s^t \int_{\mathbb{R}^3} |v|^2 (\psi_\tau + \Delta \psi) dx d\tau + \int_s^t \int_{\mathbb{R}^3} (|v|^2 + 2\pi_v) v \cdot \nabla \psi dx d\tau, \end{aligned} \quad (4.7)$$

for all $t \geq s$, for $s = 0$ and a.e. in $s \geq 0$, and for all nonnegative $\psi \in C_0^\infty(\mathbb{R}; C_b^2(\mathbb{R}^3))$.

Proof. Taking into account the integrability properties of a suitable weak solution, all the terms in (4.7) make sense. Let us define a sequence of smooth cut-off functions $\{h_\rho(x)\}$ with $h_\rho(x) \in [0, 1]$, $h_\rho(x) = 1$ for $|x| \leq \rho$, $h_\rho(x) = 0$ for $|x| \geq 2\rho$. Then for any $\eta > 0$ and $\rho > 0$ the function $\phi^{\rho, \eta}(t, x) := J_\eta(h_\rho(x)\psi(t, x))$, where J_η is a Friedrichs mollifier, belongs to $C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$. Hence, from (1.2) written with ϕ replaced by $\phi^{\rho, \eta}$, passing to the limit as $\eta \rightarrow 0$ and, subsequently, as $\rho \rightarrow \infty$, by the integrability properties of (v, π_v) and the Lebesgue dominated convergence theorem, we obtain the result. \square

Lemma 4.5 *In the hypotheses of Theorem 1.2, for all $t > 0$ and for all nonnegative function $\varphi \in C_b^2(\mathbb{R}^3)$, such that $\varphi = 0$ for $|x| \leq \max\{2R_0, 1\}$, the following inequality holds*

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^3} |u(t)|^2 \varphi dy + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \varphi dy d\tau &\leq \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |u|^2 \Delta \varphi dy d\tau + \int_0^t \int_{\mathbb{R}^3} u \cdot \nabla \varphi \pi_u dy d\tau \\ &+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \left[|u|^2 u \cdot \nabla \varphi + |u|^2 w \cdot \nabla \varphi \right] dy d\tau \\ &+ \int_0^t \int_{\mathbb{R}^3} \left[u \cdot \nabla u \cdot w \varphi + u \cdot \nabla \varphi u \cdot w + w \cdot \nabla u \cdot w \varphi + w \cdot \nabla \varphi u \cdot w \right] dy d\tau. \end{aligned} \quad (4.8)$$

Proof. The proof is an easy consequence of the Leray-Serrin technique. Indeed, from Lemma 4.4, we can consider the generalized energy inequality (4.7) for a weak solution v with $\psi(\tau, x) := h(\tau)\varphi(x)$, with $\varphi \in C_b^2(\mathbb{R}^3)$ nonnegative, such that $\varphi = 0$ for $|x| \leq \max\{2R_0, 1\}$, and $h \in [0, 1]$ smooth cut-off function such that $h(\tau) = 1$ for $\tau \in [s, t]$, $t > s > 2\varepsilon$, $h(\tau) = 0$ for $\tau < \varepsilon$ and in a neighborhood of T . Then the following inequality holds:

$$\begin{aligned} \int_{\mathbb{R}^3} |v(t)|^2 \varphi dx + 2 \int_s^t \int_{\mathbb{R}^3} |\nabla v|^2 \varphi dx d\tau &\leq \int_{\mathbb{R}^3} |v(s)|^2 \varphi dx \\ &+ \int_s^t \int_{\mathbb{R}^3} |v|^2 \Delta \varphi dx d\tau + \int_s^t \int_{\mathbb{R}^3} (|v|^2 + 2\pi_v) v \cdot \nabla \varphi dx d\tau. \end{aligned} \quad (4.9)$$

Reasoning in analogous way for w , but recalling the regularity of w and the linear character of the equations, we deduce

$$\int_{\mathbb{R}^3} |w(t)|^2 \varphi dx + 2 \int_s^t \int_{\mathbb{R}^3} |\nabla w|^2 \varphi dx d\tau = \int_{\mathbb{R}^3} |w(s)|^2 \varphi dx + \int_s^t \int_{\mathbb{R}^3} |w|^2 \Delta \varphi dx d\tau. \quad (4.10)$$

In the weak formulation of (v, π_v) we can replace the test function $\varphi(\tau, x)$ by the function $\psi(\tau, x) h_\rho(x) w(\tau, x) = h(\tau) \varphi(x) h_\rho(x) w(\tau, x)$, with $\{h_\rho(x)\} \subset [0, 1]$ sequence of smooth cut-off functions, $h_\rho(x) = 1$ for $|x| \leq \rho$, $h_\rho(x) = 0$ for $|x| \geq 2\rho$. Then for any $\rho > 0$ the function $\psi(\tau, x) h_\rho(x) w(\tau, x)$ belongs to $C_0^1([0, T] \times \mathbb{R}^3)$. Hence, we get

$$\begin{aligned} (v(t), w(t) \varphi h_\rho) + \int_s^t \int_{\mathbb{R}^3} \nabla v \cdot \nabla w \varphi h_\rho dy d\tau &= (v(s), w(s) \varphi h_\rho) \\ + \int_s^t \left[(v, w_\tau \varphi h_\rho) - (\nabla v, \nabla(\varphi h_\rho) \otimes w) + (v \cdot \nabla(\varphi h_\rho), v \cdot w) \right. \\ &\quad \left. + (v \cdot \nabla w, \varphi h_\rho v) + (\pi_v, w \cdot \nabla(\varphi h_\rho)) \right] d\tau. \end{aligned}$$

Recalling that $w_\tau = \Delta w$, and observing that, by interpolation, estimate (2.6) and the energy inequality imply $v \in L^4(0, T; L^4(\mathbb{R}^3 \setminus B_{M_0 R_0}))$, we have

$$\begin{aligned} (v(t), w(t) \varphi h_\rho) + 2 \int_s^t \int_{\mathbb{R}^3} \nabla v \cdot \nabla w \varphi h_\rho dy d\tau &= (v(s), w(s) \varphi h_\rho) \\ + \int_s^t \left[(v, w \Delta(\varphi h_\rho)) + (v \cdot \nabla(\varphi h_\rho), v \cdot w) + (v \cdot \nabla w, \varphi h_\rho v) + (\pi_v, w \cdot \nabla(\varphi h_\rho)) \right] d\tau. \end{aligned}$$

By using the integrability properties of v and w , passing to the limit as ρ tends to infinity we find

$$\begin{aligned} (v(t), w(t) \varphi) + 2 \int_s^t \int_{\mathbb{R}^3} \nabla v \cdot \nabla w \varphi dy d\tau &= (v(s), w(s) \varphi) \\ + \int_s^t \left[(v, w \Delta \varphi) + (v \cdot \nabla \varphi, v \cdot w) + (v \cdot \nabla w, \varphi v) + (\pi_v, w \cdot \nabla \varphi) \right] d\tau. \end{aligned}$$

Multiplying this last relation by -2 and summing to the inequalities (4.9)-(4.10), recalling also that φ is null for $|x| \leq \max\{2R_0, 1\}$ ensures that $w(t, x)$ satisfies estimate

(3.5), we deduce

$$\begin{aligned}
\frac{1}{2} \int_{\mathbb{R}^3} |u(t)|^2 \varphi dy + \int_s^t \int_{\mathbb{R}^3} |\nabla u|^2 \varphi dy d\tau &\leq \frac{1}{2} \int_{\mathbb{R}^3} |u(s)|^2 \varphi dy + \frac{1}{2} \int_s^t \int_{\mathbb{R}^3} |u|^2 \Delta \varphi dy d\tau \\
&+ \int_s^t \int_{\mathbb{R}^3} u \cdot \nabla \varphi \pi_u dy d\tau + \frac{1}{2} \int_s^t \int_{\mathbb{R}^3} \left[|u|^2 u \cdot \nabla \varphi + |u|^2 w \cdot \nabla \varphi \right] dy d\tau \\
&+ \int_s^t \int_{\mathbb{R}^3} \left[u \cdot \nabla u \cdot w \varphi + u \cdot \nabla \varphi u \cdot w + w \cdot \nabla u \cdot w \varphi + w \cdot \nabla \varphi u \cdot w \right] dy d\tau.
\end{aligned} \tag{4.11}$$

Thanks to the integrability properties of u and the regularity of w for $|x| > 2R_0$, given in Lemma 3.1, we can pass to the limit as $s > 2\varepsilon$ tends to 0 and we get (4.8). \square

This lemma is a relevant tool for proving Lemma 4.6 on the asymptotic behavior in R of the L^2 -norm of $u = v - w$ outside the ball B_R , uniformly in α . Actually, if we start from the energy inequality, we could get an asymptotic behavior in R only for $\alpha > \frac{3}{2}$.

Assume that $\varphi \in C_b^2(\mathbb{R}^3)$ is defined as follows

$$\begin{aligned}
m \geq 2, \quad \varphi := \varphi_R(m) &:= \begin{cases} 1 & \text{if } |x| \geq \frac{m}{m+1}R, \\ \in [0, 1] & \text{if } |x| \in [\frac{m-1}{m}R, \frac{m}{m+1}R], \\ 0 & \text{if } |x| \leq \frac{m-1}{m}R, \end{cases} \\
|\nabla \varphi_R(m)| + R|\Delta \varphi_R(m)| &\leq c(m)R^{-1}.
\end{aligned} \tag{4.12}$$

We define $\phi_R(m)(\tau, x) := k(\tau) \varphi_R(m)(x)$ with $k(\tau) \in [0, 1]$ smooth cut-off function such that $k(\tau) = 1$ for $|\tau| \leq t$ and $k(\tau) = 0$ for $|\tau| \geq 2t$. Since $\phi_R(m)$ belongs to $C_0^\infty(\mathbb{R}; C_b^2(\mathbb{R}^3))$, by Lemma 4.4 we can use $\phi_R(m)$ as test function in (1.2), and we get the following generalized energy inequality (4.13), that we will call generalized Leray's energy inequality:

$$\begin{aligned}
\int_{\mathbb{R}^3} |v(t)|^2 \varphi_R(m) dx + 2 \int_s^t \int_{\mathbb{R}^3} |\nabla v|^2 \varphi_R(m) dx d\tau &\leq \int_{\mathbb{R}^3} |v(s)|^2 \varphi_R(m) dx \\
&+ \int_s^t \int_{\mathbb{R}^3} |v|^2 \Delta \varphi_R(m) dx d\tau + \int_s^t \int_{\mathbb{R}^3} (|v|^2 + 2\pi_v) v \cdot \nabla \varphi_R(m) dx d\tau,
\end{aligned} \tag{4.13}$$

for all $t \geq s$, for $s = 0$ and a.e. in $s \geq 0$.

Lemma 4.6 *The following estimate holds*

$$\|u(t)\|_{L^2(6,R)} \leq cR^{-\frac{1}{2}}t^{\frac{1}{2}}C(v_\circ), \quad R > 4R_0, t > 0, \tag{4.14}$$

with $C^2(v_\circ) := V_\circ^4 + V_\circ^2 \|v_\circ\|_2^2 + V_\circ^2 \|v_\circ\|_2 + V_\circ \|v_\circ\|_2^2 + \|v_\circ\|_2^2 + \|v_\circ\|_2^3$.

Proof. We consider the sequence of function (4.12). In (4.8) we replace φ by $\varphi_R^2(k)$. By obvious meaning of the symbols on the right-hand side, we obtain

$$\frac{1}{2} \|u(t) \varphi_R(k)\|_2^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \varphi_R^2(k) dy d\tau \leq \sum_{i=1}^8 |I_i(t, k)|.$$

We estimate each $I_i(t, k)$, $i = 1, \dots, 8$. Recalling the definition of $\varphi_R(k)$, we get

- $|I_1| \leq c(k)R^{-2} \int_0^t \|u\|_{L^2(k-1, R)}^2 d\tau;$

- by virtue of estimate (4.4) applying Hölder's inequality, we get

$$\begin{aligned} |I_2| &\leq c(k)R^{-1} \int_0^t \|\pi_u\|_{L^1(k-1, R)} \|u\|_{L^2(k-1, R)} d\tau \leq c(k)R^{-1} \int_0^t \|\pi_u\|_{L^2(k-1, R)} \|u\|_{L^2(k-1, R)} d\tau \\ &\leq c(k)R^{-1} \int_0^t \left[\|u\|_{L^2(k-2, R)}^{\frac{1}{2}} \|\nabla u\|_{L^2(k-2, R)}^{\frac{3}{2}} d\tau + \frac{V_\circ}{R^\alpha} \|u\|_{L^2(k-2, R)} \right. \\ &\quad \left. + c \frac{V_\circ^2}{R^{2\alpha-\frac{3}{2}}} + \frac{1}{R^{\frac{3}{2}}} \|v_\circ\|_2^2 \right] \|u\|_{L^2(k-1, R)} d\tau. \end{aligned}$$

- by virtue of estimate (4.1), we get

$$|I_3| \leq c(k)R^{-1} \int_0^t \|u\|_{L^3(k-1, R)}^3 d\tau \leq c(k)R^{-1} \int_0^t \|u\|_{L^2(k-1, R)}^{\frac{3}{2}} \|\nabla u\|_{L^2(k-1, R)}^{\frac{3}{2}} d\tau;$$

- by virtue of (3.5), we have

$$|I_4 + I_6| \leq c(k)R^{-1-\alpha} V_\circ \int_0^t \|u\|_{L^2(k-1, R)}^2 d\tau;$$

- by virtue of (3.5), applying first the Hölder inequality and then the Cauchy inequality, we get

$$\begin{aligned} |I_5| &\leq cR^{-2\alpha} V_\circ^2 \int_0^t \|u\varphi_R(k)\|_2^2 d\tau + \frac{1}{4} \int_0^t \|\varphi_R(k)\nabla u\|_2^2 d\tau \\ &\leq cR^{-2\alpha} V_\circ^2 \int_0^t \|u\|_{L^2(k-1, R)}^2 d\tau + \frac{1}{4} \int_0^t \|\varphi_R(k)\nabla u\|_2^2 d\tau; \end{aligned}$$

- by virtue of (3.5), applying the Hölder inequality and then the Cauchy inequality, we get

$$\begin{aligned} |I_7| &\leq c \int_0^t \|w\varphi_R(k)\|_4^4 d\tau + \frac{1}{4} \int_0^t \|\varphi_R(k)\nabla u\|_2^2 d\tau \\ &\leq cV_\circ^4 R^{-4\alpha+3} t + \frac{1}{4} \int_0^t \|\varphi_R(k)\nabla u\|_2^2 d\tau; \end{aligned}$$

- by virtue of (3.5), applying the Hölder inequality, we get

$$|I_8| \leq c(k)R^{-1} \int_0^t \|w\|_{L^4(k-1,R)}^2 \|u\|_{L^2(k-1,R)} d\tau \leq c(k)R^{-2\alpha+\frac{1}{2}} V_\circ^2 \int_0^t \|u\|_{L^2(k-1,R)} d\tau.$$

The above estimates allow to deduce the following one:

$$\begin{aligned} & \frac{1}{2} \|u(t)\varphi_R(k)\|_2^2 + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \varphi_R^2(k) dy d\tau \\ & \leq c(k)R^{-1} \left[C_0(v_\circ)t + \int_0^t \|u\|_{L^2(k-2,R)}^{\frac{3}{2}} \|\nabla u\|_{L^2(k-2,R)}^{\frac{3}{2}} d\tau \right], \end{aligned} \quad (4.15)$$

with $C_0(v_\circ) := V_\circ^4 + V_\circ^2 \|v_\circ\|_2^2 + V_\circ^2 \|v_\circ\|_2 + V_\circ \|v_\circ\|_2^2 + \|v_\circ\|_2^2$. Writing estimate (4.15) with $k = 2$ gives

$$\frac{1}{2} \|u(t)\varphi_R(2)\|_2^2 + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \varphi_R^2(2) dy d\tau \leq c(2)R^{-1} (C_0(v_\circ) + \|v_\circ\|_2^3) t^{\frac{1}{4}}, \text{ for } t \in (0, 1), \quad (4.16)$$

and

$$\frac{1}{2} \|u(t)\varphi_R(2)\|_2^2 + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \varphi_R^2(2) dy d\tau \leq c(2)R^{-1} (C_0(v_\circ) + \|v_\circ\|_2^3) t, \text{ for } t > 1, \quad (4.17)$$

which proves (4.14) for $t \geq 1$. Taking into account estimate (4.16), we evaluate (4.15) for $k = 4$ and $t \in (0, 1)$, and we get

$$\frac{1}{2} \|u(t)\varphi_R(4)\|_2^2 + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \varphi_R^2(4) dy d\tau \leq c(4)R^{-1} t^{\frac{5}{8}} (C_0(v_\circ) + \|v_\circ\|_2^3), \text{ for } t \in (0, 1). \quad (4.18)$$

Taking into account (4.18) and evaluating (4.15) for $k = 6$ and $t \in (0, 1)$, we get

$$\frac{1}{2} \|u(t)\varphi_R(6)\|_2^2 + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \varphi_R^2(6) dy d\tau \leq c(6)R^{-1} (C_0(v_\circ)t + \|v_\circ\|_2^3 t^{\frac{19}{16}}), \text{ for } t \in (0, 1),$$

which gives (4.14) for $t \in (0, 1)$. This last estimate and (4.17) complete the proof. \square

5 Pointwise representation of the weak solution for $(t, x) \in (0, T) \times (\mathbb{R}^3 \setminus B_{2R_0})$

The following result is similar to Lemma 3.5 in [12], but we are replacing a J^2 -continuity assumption with a J^2 -weak continuity one.

Lemma 5.1 *Let $v(t)$ be a $J^2(\mathbb{R}^3)$ -weakly continuous function on $[0, T)$, $\psi \in C(0, T; J^2(\mathbb{R}^3))$, and $\psi_0 \in L^2(\mathbb{R}^3)$ such that $\lim_{t \rightarrow 0}(\psi(t), \varphi) = (\psi_0, \varphi)$ for all $\varphi \in J^{1,2}(\mathbb{R}^3)$. Then, for all $t \in (0, T)$, the following limit property holds*

$$\lim_{\delta \rightarrow 0} (v(t - \delta), \psi(\delta)) = (v(t), \psi_0).$$

Proof. From the limit assumption on $\psi(t)$ as $t \rightarrow 0$ and from the $J^2(\mathbb{R}^3)$ -strong continuity of $\psi(t)$, we can infer that $\psi(0) = P(\psi_0)$. Since for any $t \in [0, T)$, $v(t) \in J^2(\mathbb{R}^3)$, for any $\delta > 0$ we have

$$\begin{aligned} |(v(t - \delta), \psi(\delta)) - (v(t), \psi_0)| &= |(v(t - \delta) - v(t), \psi_0) + (v(t - \delta), \psi(\delta) - \psi_0)| \\ &\leq |(v(t - \delta) - v(t), \psi(0))| + \|v(t - \delta)\| \|\psi(\delta) - \psi(0)\|. \end{aligned}$$

Using the J^2 -weak continuity for the first term on the right-hand side, and the J^2 -strong continuity of $\psi(\delta)$ for second one, we get the result. \square

We premise the following regularity result of the weak solution $v(t, x)$:

Lemma 5.2 *In the hypotheses of Theorem 1.2 we get $v(t, x) \in C(0, T; L^\infty(\mathbb{R}^3 \setminus \overline{B}_{2M_0R_0}))$.*

Proof. Let $\{h_\eta\} \subset [0, 1]$ be a sequence of smooth cut-off functions with $h_\eta(t - \tau) = 1$ for $t - \tau > 2\eta$, $h_\eta(t - \tau) = 0$ for $t - \tau < \eta$. Let us consider the Navier-Stokes weak formulation corresponding to a solution (v, π_v) written on the interval $(0, t - 2\eta)$, with $h_\eta(t - \tau)\varphi_R(4)\theta^t(\tau, x)$ as test function. The function $\varphi_R(4)$ is defined in (4.12), and we set $R := 2M_0R_0$, while $\theta^t := \theta(t - \tau, x)$, for $\tau \in (0, t)$, with $\theta(\sigma, x)$ solution to the Stokes Cauchy problem (3.1) given in Lemma 3.2. It is known that θ^t is a backward in time solution to the Stokes Cauchy problem on $(0, t) \times \mathbb{R}^3$. In the following, since there is no danger confusion, we denote $\varphi_R(4)$ simply by φ .

Hence, after substituting, we get

$$\begin{aligned} (v(t - 2\eta), \varphi\theta(2\eta)) &= (v_0, \varphi\theta(t)) + 2 \int_0^{t-2\eta} (v, \nabla\varphi \cdot \nabla\theta^t) d\tau + \int_0^{t-2\eta} (v, \theta^t \Delta\varphi) d\tau \\ &\quad - \int_0^{t-2\eta} (v \cdot \nabla v, \varphi\theta^t) d\tau + \int_0^{t-2\eta} (\pi_v, \nabla\varphi \cdot \theta^t) d\tau. \end{aligned} \tag{5.1}$$

The same relation written on the interval $(0, s - 2\eta)$ and with the test function $h_\eta(s - \tau)\varphi_R(4)\theta^s(\tau, x)$ furnishes

$$\begin{aligned} (v(s - 2\eta), \varphi\theta(2\eta)) &= (v_0, \varphi\theta(s)) + 2 \int_0^{s-2\eta} (v, \nabla\varphi \cdot \nabla\theta^s) d\tau + \int_0^{s-2\eta} (v, \theta^s \Delta\varphi) d\tau \\ &\quad - \int_0^{s-2\eta} (v \cdot \nabla v, \varphi\theta^s) d\tau + \int_0^{s-2\eta} (\pi_v, \nabla\varphi \cdot \theta^s) d\tau. \end{aligned} \tag{5.2}$$

From Lemma 5.1 one has

$$\lim_{\eta \rightarrow 0} (v(t - 2\eta), \theta(2\eta)) = (v(t), \theta_0), \text{ and } \lim_{\eta \rightarrow 0} (v(s - 2\eta), \theta(2\eta)) = (v(s), \theta_0).$$

Passing to the limit as η tends to 0 on the right-hand side of (5.1) and (5.2), we can use the Lebesgue dominated convergence theorem, observing that the integrals on $(0, t)$, such as the integrals on $(0, s)$, are finite thanks to the estimates (3.6) with $q < \frac{3}{2}$. Hence, we get

$$\begin{aligned} (v(t), \varphi\theta_0) &= (v_\circ, \varphi\theta(t)) + 2 \int_0^t (v, \nabla\varphi \cdot \nabla\theta^t) d\tau + \int_0^t (v, \theta^t \Delta\varphi) d\tau \\ &\quad - \int_0^t (v \cdot \nabla v, \varphi\theta^t) d\tau + \int_0^t (\pi_v, \nabla\varphi \cdot \theta^t) d\tau, \end{aligned}$$

and

$$\begin{aligned} (v(s), \varphi\theta_0) &= (v_\circ, \varphi\theta(s)) + 2 \int_0^s (v, \nabla\varphi \cdot \nabla\theta^s) d\tau + \int_0^s (v, \theta^s \Delta\varphi) d\tau \\ &\quad - \int_0^s (v \cdot \nabla v, \varphi\theta^s) d\tau + \int_0^s (\pi_v, \nabla\varphi \cdot \theta^s) d\tau. \end{aligned}$$

Then, we deduce

$$\begin{aligned} (v(t) - v(s), \varphi\theta_0) &= (v_\circ, \varphi(\theta(t) - \theta(s))) + 2 \int_s^t (v, \nabla\varphi \cdot \nabla\theta^t) d\tau + \int_s^t (v, \theta^t \Delta\varphi) d\tau \\ &\quad - \int_s^t (v \cdot \nabla v, \varphi\theta^t) d\tau + \int_s^t (\pi_v, \nabla\varphi \cdot \theta^t) d\tau \\ &\quad + \sum_{i=1}^3 \left[2 \int_{s_i}^{s_{i+1}} (v, \nabla\varphi \cdot (\nabla\theta^t - \nabla\theta^s)) d\tau + \int_{s_i}^{s_{i+1}} (v, (\theta^t - \theta^s) \Delta\varphi) d\tau \right. \\ &\quad \left. - \int_{s_i}^{s_{i+1}} (v \cdot \nabla v, \varphi(\theta^t - \theta^s)) d\tau + \int_{s_i}^{s_{i+1}} (\pi_v, \nabla\varphi \cdot (\theta^t - \theta^s)) d\tau \right], \end{aligned} \tag{5.3}$$

with $s_1 = 0, s_2 = \varepsilon, s_3 = s - \varepsilon$ and $s_4 = s$. We limit ourselves to estimate the first term, all the integrals involving the nonlinear term $v \cdot \nabla v$ and the pressure field, as the others are of simpler discussion.

- i) Applying Hölder's inequality and the semigroup properties (3.7) of the solution θ , we get

$$|(v_\circ, \varphi(\theta(t) - \theta(s)))| \leq \|v_\circ\|_2 \|\theta(t) - \theta(s)\|_2 \leq c \|v_\circ\|_2 \xi^{-1-\frac{3}{4}}(t-s) \|\theta_0\|_1;$$

- ii) applying Hölder's inequality and the semigroup properties (3.6), recalling estimate

(2.6) for the weak solution v , for $p = 6$, we get

$$\begin{aligned}
\left| \int_s^t (v \cdot \nabla v, \varphi \theta^t) d\tau \right| &\leq \int_s^t [\| |v|^2 \nabla \varphi | \theta^t \|_1 + \| |v|^2 \varphi | \nabla \theta^t \|_1] d\tau \\
&\leq \int_s^t [\| v \|_{L^{2p}(4,R)}^2 \| \theta^t \|_{p'} + \| v \|_{L^{2p}(4,R)}^2 \| \nabla \theta^t \|_{p'}] d\tau \\
&\leq \int_s^t [\| v \|_{L^2(4,R)}^{\frac{2}{p}} \| v \|_{L^\infty(4,R)}^{\frac{2}{p'}} (\| \theta^t \|_{p'} + \| \nabla \theta^t \|_{p'})] d\tau \\
&\leq \| v_o \|_2^{\frac{1}{3}} c(v_o)^{\frac{5}{3}} s^{-\frac{5}{6}} [(t-s)^{\frac{3}{4}} + (t-s)^{\frac{1}{4}}] \| \theta_0 \|_1 ;
\end{aligned}$$

iii) applying Hölder's inequality, the semigroup properties (3.6) of θ and taking into account that $t - \tau > s - \tau$, we obtain

$$\begin{aligned}
\left| \int_{s_1}^{s_2} (v \cdot \nabla v, \varphi(\theta^t - \theta^s)) d\tau \right| &\leq \int_{s_1}^{s_2} \| v \cdot \nabla v \theta^t \|_1 + \| v \cdot \nabla v \theta^s \|_1 d\tau \\
&\leq \int_{s_1}^{s_2} \| v \|_2 \| \nabla v \|_2 \| \theta^t \|_\infty + \| v \|_2 \| \nabla v \|_2 \| \theta^s \|_\infty d\tau \\
&\leq c \| v_o \|_2 \left[\int_0^\varepsilon (s - \tau)^{-3} d\tau \right]^{\frac{1}{2}} \left[\int_0^\varepsilon \| \nabla v \|_2^2 d\tau \right]^{\frac{1}{2}} \| \theta_0 \|_1 \leq c \| v_o \|_2^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \frac{(2s - \varepsilon)^{\frac{1}{2}}}{s(s - \varepsilon)} \| \theta_0 \|_1 ;
\end{aligned}$$

iv) applying Hölder's inequality and the semigroup properties (3.7) of θ , recalling estimate (2.6) for v and that $s - \tau < t - \tau$, for $p = 6$, we get

$$\begin{aligned}
\left| \int_{s_2}^{s_3} (v \cdot \nabla \varphi, v \cdot (\theta^t - \theta^s)) d\tau \right| + \left| \int_{s_2}^{s_3} (v \cdot \nabla (\theta^t - \theta^s), v \varphi) d\tau \right| \\
\leq \int_{s_2}^{s_3} [\| v \|_{2p}^2 \| \nabla (\theta^t - \theta^s) \|_{p'} + \| v \|_{2p}^2 \| \theta^t - \theta^s \|_{p'}] d\tau \\
\leq c \| v_o \|_2^{\frac{2}{p}} c(v_o)^{\frac{2}{p'}} \| \theta_0 \|_1 (t - s) \int_\varepsilon^{s-\varepsilon} \frac{\xi^{-\frac{3}{2} - \frac{3}{2p}} + \xi^{-1 - \frac{3}{2p}}}{\tau^{\frac{1}{p'}}} d\tau \\
\leq c \| v_o \|_2^{\frac{1}{3}} c(v_o)^{\frac{5}{3}} [\varepsilon^{-\frac{19}{12}} + \varepsilon^{-\frac{13}{12}}] (t - s) \| \theta_0 \|_1 ;
\end{aligned}$$

v) applying Hölder's inequality and the semigroup properties (3.6), recalling estimate

(2.6) for the weak solution v and that $s - \tau < t - \tau$, for $p = 6$, we get

$$\begin{aligned}
& \left| \int_{s_3}^{s_4} (v \cdot \nabla \varphi, v \cdot (\theta^t - \theta^s)) d\tau \right| + \left| \int_{s_3}^{s_4} (v \cdot \nabla (\theta^t - \theta^s), v \varphi) d\tau \right| \\
& \leq \int_{s_3}^{s_4} \|v\|_{L^{2p}(4)}^2 \|\nabla(\theta^t - \theta^s)\|_{p'} + \|v\|_{L^{2p}(4)}^2 \|\theta^t - \theta^s\|_{p'} d\tau \\
& \leq c \|v_\circ\|_2^{\frac{2}{p}} c(v_\circ)^{\frac{2}{p'}} \left[\int_{s-\varepsilon}^s \frac{\|\nabla \theta^t\|_{p'} + \|\nabla \theta^s\|_{p'} + \|\theta^t\|_{p'} + \|\theta^s\|_{p'}}{\tau^{\frac{1}{p'}}} d\tau \right] \\
& \leq c \|v_\circ\|_2^{\frac{1}{3}} c(v_\circ)^{\frac{5}{3}} \frac{\varepsilon^{\frac{1}{4}} + \varepsilon^{\frac{3}{4}}}{(s - \varepsilon)^{\frac{5}{6}}} \|\theta_0\|_1.
\end{aligned}$$

Finally, we estimate the terms with the pressure field π_v . To this end, taking into account that, by definition, $\nabla \varphi$ is nonnull on $B_{\frac{4}{3}R} \setminus B_{\frac{3}{4}R}$, we can use estimate (4.6) for $k = 3$:

$$\|\pi_v\|_{L^2(3,R)} \leq c(\|v\|_2^2 + \|v\|_{L^4(2,R)}^2), \text{ a. e. in } t > 0.$$

By using (2.6) and the energy relation, we get

$$\|\pi_v\|_{L^2(3,R)} \leq c(\|v_\circ\|_2^2 + \|v_\circ\|_2 c(v_\circ) t^{-\frac{1}{2}}). \quad (5.4)$$

We start with the estimate on (s, t) :

j) applying Hölder's inequality, the semigroup properties (3.6) for θ and estimate (5.4), we deduce

$$\left| \int_s^t (\pi_v \nabla \varphi, \theta^t) d\tau \right| \leq \int_s^t \|\pi_v\|_{L^2(3,R)} \|\theta^t\|_2 d\tau \leq c(\|v_\circ\|_2^2 + \|v_\circ\|_2 c(v_\circ) s^{-\frac{1}{2}}) (t - s)^{\frac{1}{4}} \|\theta_0\|_1;$$

jj) by the same arguments as before and taking into account that $s - \tau < t - \tau$, we get

$$\begin{aligned}
\left| \int_{s_1}^{s_2} (\pi_v, \nabla \varphi \cdot (\theta^t - \theta^s)) d\tau \right| & \leq c \int_0^\varepsilon \|\pi_v\|_{L^2(3,R)} (\|\theta^t\|_2 + \|\theta^s\|_2) d\tau \\
& \leq \frac{c}{(s - \varepsilon)^{\frac{3}{4}}} \|\theta_0\|_1 \int_0^\varepsilon (\|v_\circ\|_2^2 + \|v_\circ\|_2 c(v_\circ) \tau^{-\frac{1}{2}}) d\tau \\
& \leq \frac{c}{(s - \varepsilon)^{\frac{3}{4}}} (\|v_\circ\|_2^2 \varepsilon^{\frac{1}{2}} + \|v_\circ\|_2 c(v_\circ)) \varepsilon^{\frac{1}{2}} \|\theta_0\|_1;
\end{aligned}$$

jjj) by the same arguments and employing estimate (3.7), we get

$$\begin{aligned}
\left| \int_{s_2}^{s_3} (\pi_v, \nabla \varphi \cdot (\theta^t - \theta^s)) d\tau \right| &\leq c \int_{\varepsilon}^{s-\varepsilon} \|\pi_v\|_{L^2(3,R)} \|\theta^t - \theta^s\|_2 d\tau \\
&\leq c(\|v_o\|_2^2 + \|v_o\|_2 c(v_o) \varepsilon^{-\frac{1}{2}}) \int_{\varepsilon}^{s-\varepsilon} \|\theta^t - \theta^s\|_2 d\tau \\
&\leq c(\|v_o\|_2^2 + \|v_o\|_2 c(v_o) \varepsilon^{-\frac{1}{2}}) \varepsilon^{-\frac{7}{4}} (t-s) \|\theta_0\|_1;
\end{aligned}$$

ju) finally,

$$\begin{aligned}
\left| \int_{s_3}^{s_4} (\pi_v, \nabla \varphi \cdot (\theta^t - \theta^s)) d\tau \right| &\leq c \int_{s-\varepsilon}^s \|\pi_v\|_{L^2(3,R)} \|\theta^t - \theta^s\|_2 d\tau \\
&\leq c(\|v_o\|_2^2 + \|v_o\|_2 c(v_o) (s-\varepsilon)^{-\frac{1}{2}}) \int_{s-\varepsilon}^s (\|\theta^t\|_2 + \|\theta^s\|_2) d\tau \\
&\leq c(\|v_o\|_2^2 + \|v_o\|_2 c(v_o) (s-\varepsilon)^{-\frac{1}{2}}) \varepsilon^{\frac{1}{4}} \|\theta_0\|_1.
\end{aligned}$$

Considering estimates i)-v) and j)-ju), and the corresponding estimates for the linear terms of relation (5.3), we deduce

$$|(v(t) - v(s), \varphi \theta_0)| \leq F(\varepsilon, s, t, v_o) \|\theta_0\|_1, \text{ for arbitrary } \theta_0 \in C_0(\mathbb{R}^3),$$

with clear meaning of function F , hence

$$\|(v(t) - v(s))\varphi\|_{\infty} \leq F(\varepsilon, s, t, v_o).$$

Since we can assume $t - s < \varepsilon^4$, for fixed $s > 0$ and v_o the above estimates ensure $\lim_{\varepsilon \rightarrow 0} F(\varepsilon, s, t, v_o) = 0$, the lemma is proved. \square

In the following lemma we give the representation formula for the weak solution (v, π_v) provided that $|x| > 2M_0 R_0$. To this end, we recall that the representation formula is given by means of the fundamental heat kernel H (for its definition and properties see (3.2)-(3.3)) and the Oseen tensor T (see [7]):

$$\begin{aligned}
T_{ij}(t-\tau, x-y) &:= -\Delta \phi(t-\tau, |x-y|) \delta_{ij} + D_{x_i x_j}^2 \phi(t-\tau, |x-y|) \\
&= H_{ij}(t-\tau, x-y) + D_{x_i x_j}^2 \phi(t-\tau, |x-y|), \\
\phi(t, r) &= \frac{1}{2} \frac{1}{\sqrt{\pi^3}} \frac{1}{r} \int_0^{r/2\sqrt{t}} e^{-\rho^2} d\rho.
\end{aligned}$$

We denote by $T_j(t-\tau, x-y)$ the j -th column of the matrix T_{ij} . The pair $(T_j(t-\tau, x-y), p)$, with $p = 0$, for $t - \tau > 0$ is a solution in the (t, x) variables of the Stokes system, and in the (τ, y) variables of its adjoint system:

$$\hat{w}_\tau + \Delta \hat{w} + \nabla p = 0, \quad \nabla \cdot \hat{w} = 0.$$

The following estimate holds:

$$|D_s^k D_z^\beta T(s, z)| \leq c(|z| + s^{\frac{1}{2}})^{-3-2k-|\beta|}. \quad (5.5)$$

Finally, we recall that from the definition of T we get

$$(T_j(t, x), \varphi) = (H_j(t, x), \varphi), \text{ for all } \varphi \in J^p(\mathbb{R}^3), p > 1.$$

We set

$$\mathbb{T}[v \otimes v](s, t, x) := \int_s^t (v \cdot \nabla T_i(t, x), v) d\tau.$$

If $s = 0$, we simply write $\mathbb{T}[v \otimes v](t, x)$.

Lemma 5.3 *Let (v, π_v) be a suitable weak solution to the Navier-Stokes equations. Then, for all $t > 0$ and $s \geq 0$, and almost a.e. in $x \in \mathbb{R}^3 \setminus B_{2M_0 R_0}$,*

$$v(t, x) = \mathbb{H}[v(s)](t - s, x) + \mathbb{T}[v \otimes v](s, t, x). \quad (5.6)$$

Moreover, for $t > T_0$ and for $x \in \mathbb{R}^3$,

$$v(t, x) = \mathbb{H}[v(T_0)](t - T_0, x) + \mathbb{T}[v \otimes v](T_0, t, x). \quad (5.7)$$

Proof. Let us consider the weak formulation with a divergence free test function:

$$(v(t), \varphi(t)) = (v(s), \varphi(s)) - \int_s^t (v, \varphi_\tau + \Delta \varphi) d\tau + \int_s^t (v \cdot \nabla \varphi, v) d\tau. \quad (5.8)$$

We set, for all $i = 1, 2, 3$, and $t - \varepsilon > s$,

$$\varphi(\tau, y) := h^\varepsilon(\tau) J_n[T_i(t - \tau, x)](y), \text{ with } h^\varepsilon(\tau) := \begin{cases} 1 & \text{if } \tau \leq t - \varepsilon, \\ \in [0, 1] & \text{if } \tau \in [t - \varepsilon, t - \frac{\varepsilon}{4}], \\ 0 & \text{if } \tau > t - \frac{\varepsilon}{4}, \end{cases}$$

where J_n is a Friedrichs mollifier. Hence, inserting such a φ in (5.8) with t replaced by $t - \varepsilon$, and recalling that T_i is a solution backward in time with respect to $(\tau, y) \in (0, t) \times \mathbb{R}^3$, we obtain

$$(v_i(t - \varepsilon), J_n[H(\varepsilon, x)]) = (v_i(s), J_n[H(t - s, x)]) + \int_s^{t-\varepsilon} (v \cdot \nabla J_n[T_i(t - \tau, x)], v) d\tau. \quad (5.9)$$

We perform the limit as $\varepsilon \rightarrow 0$. To this end, we deal separately with the terms of the last integral equation.

i) The first term can be written as:

$$\begin{aligned}
& (v_i(t-\varepsilon), J_n[H(\varepsilon, x)]) - J_n[v_i(t)](x) \\
&= \int_{\mathbb{R}^3} v_i(t-\varepsilon, y) \int_{\mathbb{R}^3} J_n(y-z) H(\varepsilon, x-z) dz dy - \int_{\mathbb{R}^3} J_n(x-y) v_i(t, y) dy \\
&= \int_{\mathbb{R}^3} H(\varepsilon, x-z) \left[\int_{\mathbb{R}^3} J_n(y-z) v_i(t-\varepsilon, y) dy - \int_{\mathbb{R}^3} J_n(x-y) v_i(t, y) dy \right] dz \\
&= \int_{|x-z|>\eta} H(\varepsilon, x-z) \left[\int_{\mathbb{R}^3} J_n(y-z) v_i(t-\varepsilon, y) dy - \int_{\mathbb{R}^3} J_n(x-y) v_i(t, y) dy \right] dz \\
&\quad + \int_{|x-z|<\eta} H(\varepsilon, x-z) \int_{\mathbb{R}^3} J_n(y-z) [v_i(t-\varepsilon, y) - v_i(t, y)] dy dz \\
&\quad + \int_{|x-z|<\eta} H(\varepsilon, x-z) \int_{\mathbb{R}^3} [J_n(y-z) - J_n(x-y)] v_i(t, y) dy dz = \sum_{i=1}^3 I_i(\varepsilon, \eta).
\end{aligned}$$

Thanks to the energy inequality, the L^2 -norm of v is finite for all $t > 0$. Hence, for all $n \in \mathbb{N}$, $\left[\int_{\mathbb{R}^3} J_n(y-z) v_i(t-\varepsilon, y) dy - \int_{\mathbb{R}^3} J_n(x-y) v_i(t, y) dy \right]$ belongs to $L^\infty(\mathbb{R}^3)$, uniformly with respect to ε . Therefore it is easy to deduce that, for all $x \in \mathbb{R}^3$ and $\eta > 0$,

$$\lim_{\varepsilon \rightarrow 0} I_1(\varepsilon, \eta) = 0.$$

For the term I_2 , for $\eta > 0$ sufficiently small and n sufficiently large, we get

$$\begin{aligned}
|I_2(\varepsilon, \eta)| &\leq \|H(\varepsilon, x)\|_{L^1(B(x, \eta))} \|J_n[v_i(t-\varepsilon) - v_i(t)]\|_\infty \\
&\leq \|H(\varepsilon, x)\|_{L^1(B(x, \eta))} \|v_i(t-\varepsilon) - v_i(t)\|_{L^\infty(|y|>2M_0R_0)},
\end{aligned}$$

and, by the continuity proved in Lemma 5.2, we deduce, for all $|x| > 2M_0R_0$ and $\eta > 0$,

$$\lim_{\varepsilon \rightarrow 0} I_2(\varepsilon, \eta) = 0.$$

Finally, there holds

$$\begin{aligned}
|I_3(\varepsilon, \eta)| &\leq \|H(\varepsilon, x)\|_{L^1(B(x, \eta))} \|J_n[v_i(t)](z) - J_n[v_i(t)](x)\|_{C(B(x, \eta))} \\
&\leq \|J_n[v_i(t)](z) - J_n[v_i(t)](x)\|_{C(B(x, \eta))}, \text{ for all } \varepsilon > 0.
\end{aligned}$$

Since, for all $n \in \mathbb{N}$, $J_n[v_i(t)](z)$ is a continuous function of z , we deduce that

$$|\lim_{\eta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} I_3(\varepsilon, \eta)| \leq \lim_{\eta \rightarrow 0} \|J_n[v_i(t)](z) - J_n[v_i(t)](x)\|_{C(B(x, \eta))} = 0.$$

Hence, for all $n \in \mathbb{N}$, we have proved that

$$\lim_{\varepsilon \rightarrow 0} (v_i(t-\varepsilon), J_n[H(\varepsilon, x)]) = J_n[v_i(t)](x). \quad (5.10)$$

ii) Trivially, the second term admits limit for $\varepsilon \rightarrow 0$.

iii) For the last term it is enough to note that, for all $n \in \mathbb{N}$, $\nabla J_n[T_i(t - \tau, x)] \in L^1(0, t; L^\infty(\mathbb{R}^3))$ and $v \in L^\infty(0, T; L^2(\mathbb{R}^3))$, then, applying the Lebesgue dominated convergence theorem, we deduce the limit property

$$\lim_{\varepsilon \rightarrow 0} \int_s^{t-\varepsilon} (v \cdot \nabla J_n[T_i(t - \tau, x)], v) d\tau = \int_s^t (v \cdot \nabla J_n[T_i(t - \tau, x)], v) d\tau. \quad (5.11)$$

So that from (5.9), via (5.10)-(5.11), we have proved

$$J_n[v_i(t)](x) = \mathbb{H}[J_n[v_i(s)]](t - s, x) + \int_s^t (v \cdot \nabla J_n[T_i(t - \tau, x)], v) d\tau, \quad (5.12)$$

for $(t, x) \in (0, T) \times (\mathbb{R}^3 \setminus B_{2M_0 R_0})$.

Now, we perform the limit as $n \rightarrow \infty$. We begin by remarking that, for all t and s , $\{J_n[v(t)]\}$ and $\{J_n[v(s)]\}$ converge in $L^2(\mathbb{R}^3)$. So that there exists a subsequence, labeled again by n , converging almost everywhere in (t, x) to $v(t, x)$, and to $v(s, x)$ in $L^2(\mathbb{R}^3)$. Therefore, recalling that, for all $t - s > 0$, $H(t - s, x - y) \in L^2(\mathbb{R}^3)$, almost everywhere in $x \in \mathbb{R}^3 \setminus B_{2M_0 R_0}$ the following limit properties hold:

$$\lim_n J_n[v_i(t)](x) = v_i(t, x) \quad \text{and} \quad \lim_n \mathbb{H}[J_n[v_i(s)]](t - s, x) = \mathbb{H}[v_i(s)](t - s, x). \quad (5.13)$$

Since for $|x| > 2M_0 R_0$

$$\lim_n J_n[\nabla T_i(t - \tau, x)](y) = \nabla T_i(t - \tau, x - y) \quad \text{and} \quad |\nabla T_i(t - \tau, x - y)| \leq C(x),$$

for all $(\tau, y) \in (0, t) \times \overline{B}_{2M_0 R_0}$,

and $v \in L^\infty(0, T; L^2(\mathbb{R}^3))$, the following limit holds for all $|x| > 2M_0 R_0$:

$$\begin{aligned} \lim_n \int_s^t \int_{|y| < 2M_0 R_0} v(\tau, y) \cdot J_n[\nabla T_i(x, t - \tau)](y) \cdot v(\tau, y) dy d\tau \\ = \int_s^t \int_{|y| < 2M_0 R_0} v(\tau, y) \cdot \nabla T_i(x - y, t - \tau) \cdot v(\tau, y) dy d\tau. \end{aligned} \quad (5.14)$$

Since $\nabla T(s, z) \in L^1(0, T; L^1(\mathbb{R}^3))$, then $J_n[\nabla T_i(t - \tau, x)](y)$ converges to $\nabla T_i(t - \tau, x - y)$ in $L^1(0, T; L^1(\mathbb{R}^3))$. Moreover, since by (2.6)

$$\|v(t)\|_{L^\infty(|y| > 2M_0 R_0)} \leq c(v_o) t^{-\frac{1}{2}},$$

then the following limit holds for all $|x| > 2M_0 R_0$:

$$\begin{aligned} \lim_n \int_s^t \int_{|y| > 2M_0 R_0} v(\tau, y) \cdot J_n[\nabla T_i(t - \tau, x)](y) \cdot v(\tau, y) dy d\tau \\ = \int_s^t \int_{|y| > 2M_0 R_0} v(\tau, y) \cdot \nabla T_i(t - \tau, x - y) \cdot v(\tau, y) dy d\tau. \end{aligned} \quad (5.15)$$

So that passing to the limit in (5.12), thanks (5.13)–(5.15), we get

$$v_i(t, x) = \mathbb{H}[v_i(s)](t - s, x) + \int_s^t (v \cdot \nabla T_i(t, x), v) d\tau, \quad (5.16)$$

for $(t, x) \in (0, T) \times (\mathbb{R}^3 \setminus B_{2M_0 R_0})$.

This proves (5.6) for $s > 0$. Let us show its validity for $s = 0$. Since for $t > 0$ a solution to the heat equation is a continuous function, and since the weak solution v is a continuous function in $s = 0$ in the L^2 -norm, we easily get

$$\begin{aligned} \lim_{s \rightarrow 0} \mathbb{H}[v_i(s)](t - s, x) &= \lim_{s \rightarrow 0} \mathbb{H}[v_{\circ i}](t - s, x) + \lim_{s \rightarrow 0} \mathbb{H}[v_i(s) - v_{\circ i}](t - s, x) \\ &= \mathbb{H}[v_{\circ i}](t, x), \quad \text{a.e. in } x \in \mathbb{R}^3. \end{aligned} \quad (5.17)$$

For the integral term we have to verify that the integral is well posed on $(0, t) \times \mathbb{R}^3$. Noting that

$$|x| > 2M_0 R_0 \Rightarrow |\nabla T_i(t - \tau, x - y)| \leq c(x), \quad \text{for all } (\tau, y) \in (0, t) \times B_{2M_0 R_0},$$

we deduce

$$\int_{|y| < 2M_0 R_0} |v \cdot \nabla T_i(t, x) \cdot v| dy \leq c(x) \|v\|_2^2 \leq c \|v_{\circ}\|_2^2, \quad \text{for all } \tau \in (0, t).$$

Moreover, by virtue of (2.6) we have

$$\|v(t)\|_{L^{\frac{3-2\varepsilon}{1-2\varepsilon}}(|y| > 2M_0 R_0)}^2 \leq \|v(t)\|_2^{2\frac{1-2\varepsilon}{3-2\varepsilon}} \|v(t)\|_{L^\infty(|y| > 2M_0 R_0)}^{\frac{4}{3-2\varepsilon}} \leq c \|v_{\circ}\|_2^{2\frac{1-2\varepsilon}{3-2\varepsilon}} c(v_{\circ})^{\frac{4}{3-2\varepsilon}} t^{-\frac{2}{3-2\varepsilon}}.$$

Therefore, applying Höder's inequality, we deduce

$$\int_{|y| > 2M_0 R_0} |v \cdot \nabla T_i(t, x) \cdot v| dy \leq \|\nabla T_i(t, x)\|_{\frac{3}{2}-\varepsilon} \|v\|_{L^{\frac{3-2\varepsilon}{6-4\varepsilon}}(|y| > 2M_0 R_0)}^2 \leq c(t - \tau)^{-\frac{3-4\varepsilon}{3-2\varepsilon}} \tau^{-\frac{2}{3-2\varepsilon}}.$$

The above estimates and the limit property (5.17) allow to make the limit as $s \rightarrow 0$ in (5.16), and complete the proof of estimate (5.6).

Finally, since the solution is regular for $t > T_0$ (see [9]), we could repeat the above argument lines with obvious simplifications, starting from (5.9), and get (5.7). \square

6 Spatial behavior of the weak solution: proof of Theorem 1.2

For all $a, b < 1$, we set

$$A := \int_0^1 \tau^{-a} (1 - \tau)^{-b} d\tau.$$

Hence, we get

$$\int_0^t \tau^{-a} (t - \tau)^{-b} d\tau = A t^{1-a-b}, \quad \text{for all } t > 0. \quad (6.1)$$

To prove Theorem 1.2, as a first step, we only prove estimate (1.3) on the interval $(0, 1)$ and $|x| > R_1 := \frac{14}{3} M_0 R_0$.

Our starting point is the representation formula (5.6), that we write as follows

$$v(t, x) = w(t, x) + \mathbb{T}[u \otimes u](t, x) + \mathbb{T}[u \otimes w](t, x) + \mathbb{T}[w \otimes u](t, x) + \mathbb{T}[w \otimes w](t, x), \quad (6.2)$$

where w is the solution of the Stokes problem whose properties are established in Lemma 3.1, and $u = v - w$. Recall that, thanks to Lemma 4.6, u satisfies estimate (4.14). We introduce the following decomposition:

$$\mathbb{T}[a \otimes b](t, x) := \mathbb{T}^{(1)}[a \otimes b](t, x) + \mathbb{T}^{(2)}[a \otimes b](t, x), \quad (6.3)$$

where

$$\begin{aligned} \mathbb{T}^{(1)}[a \otimes b](t, x) &:= \int_0^t \int_{|y| < \frac{6}{7}|x|} a(\tau, y) \otimes b(\tau, y) \cdot \nabla T(t - \tau, x - y) dy d\tau, \\ \mathbb{T}^{(2)}[a \otimes b](t, x) &:= \int_0^t \int_{|y| > \frac{6}{7}|x|} a(\tau, y) \otimes b(\tau, y) \cdot \nabla T(t - \tau, x - y) dy d\tau. \end{aligned}$$

We estimate the terms on the right-hand side of (6.2). Our task is to prove that all the terms satisfy the bound given in (1.3).

- i) For the solution w estimate (1.3) follows from Lemma 3.1.
- ii) Since $|y| < \frac{6}{7}|x|$, if a and $b \in L^\infty(0, T; L^2(\mathbb{R}^3))$, employing estimate (5.5), we easily deduce

$$|\mathbb{T}^{(1)}(t, x)| \leq c \int_0^t \int_{|y| < \frac{6}{7}|x|} (|x - y| + (t - \tau)^{\frac{1}{2}})^{-4} |a||b| dy d\tau \leq c|x|^{-3} t^{\frac{1}{2}} \sup_{(0, t)} \|a\|_2 \|b\|_2.$$

Since u and $w \in L^\infty(0, T; L^2(\mathbb{R}^3))$ and $\alpha \in [1, 3)$, the above estimate ensures that

$$\begin{aligned} &|\mathbb{T}^{(1)}[u \otimes u]| + |\mathbb{T}^{(1)}[u \otimes w](t, x)| + |\mathbb{T}^{(1)}[w \otimes u](t, x)| + |\mathbb{T}^{(1)}[w \otimes w](t, x)| \\ &\leq c(v_\circ)(1 + |x|)^{-\alpha}, \quad (t, x) \in (0, 1) \times \mathbb{R}^3 \setminus B_{R_1}. \end{aligned}$$

- iii) First of all, we note that

$$|\mathbb{T}^{(2)}[u \otimes w](t, x) + \mathbb{T}^{(2)}[w \otimes u](t, x)| \leq 2 \int_0^t \int_{|y| > \frac{6}{7}|x|} (|x - y| + (t - \tau)^{\frac{1}{2}})^{-4} |u||w| dy d\tau.$$

Also, since $|y| > \frac{6}{7}|x| > 4M_0 R_0$, then (2.6) and (3.5) imply in particular

$$|u(t, y)| \leq c(v_\circ) t^{-\frac{1}{2}}. \quad (6.4)$$

Hence, from the above integral inequality and again using (3.5) for w , recalling (6.1), it follows that

$$\begin{aligned} & |\mathbb{T}^{(2)}[u \otimes w](t, x)| + |\mathbb{T}^{(2)}[w \otimes u](t, x)| + |\mathbb{T}^{(2)}[w \otimes w](t, x)| \\ & \leq 2c(v_o)(1 + |x|)^{-\alpha} \int_0^t \int_{|y| > \frac{6}{7}|x|} (|x - y| + (t - \tau)^{\frac{1}{2}})^{-4} \tau^{-\frac{1}{2}} dy d\tau = c(v_o)(1 + |x|)^{-\alpha}, \\ & (t, x) \in (0, T) \times \mathbb{R}^3 \setminus B_{R_1}. \end{aligned}$$

iv) In this step, we give a first estimate for $\mathbb{T}^{(2)}[u, u](t, x)$ of the kind (1.3). Subsequently, employing all together estimates i)-iv), we prove (1.3) for $\mathbb{T}^{(2)}[u, u](t, x)$, completing the proof of the theorem for $t \in (0, 1)$.

First of all, using the interpolation between L^q -spaces, estimate (4.14) and estimate (6.4) give

$$\|u\|_{L^{\frac{6-4\varepsilon}{1-2\varepsilon}}(6, |x|)} \leq \|u\|_{L^2(6, |x|)}^{\frac{1-2\varepsilon}{3-2\varepsilon}} \|u\|_{L^\infty(6, |x|)}^{\frac{2}{3-2\varepsilon}} \leq c(v_o) t^{-\frac{1}{2} \frac{1+2\varepsilon}{3-2\varepsilon}} |x|^{-\frac{1}{2} \frac{1-2\varepsilon}{3-2\varepsilon}}, t > 0, \quad (6.5)$$

where, here and in the following, $\varepsilon \in (0, \frac{1}{2})$. Applying Hölder's inequality, then estimates (6.5) and (6.1), we get

$$\begin{aligned} |\mathbb{T}^{(2)}[u, u](t, x)| & \leq c \int_0^t \|\nabla T(t, x)\|_{\frac{3}{2}-\varepsilon} \|u\|_{L^{\frac{6-4\varepsilon}{1-2\varepsilon}}(|y| > \frac{6}{7}|x|)}^2 d\tau \\ & \leq c(v_o) \int_0^t (t - \tau)^{-\frac{3-4\varepsilon}{3-2\varepsilon}} \tau^{-\frac{1+2\varepsilon}{3-2\varepsilon}} d\tau \leq c(v_o) t^{-\frac{1}{3-2\varepsilon}} |x|^{-\frac{1-2\varepsilon}{3-2\varepsilon}}. \end{aligned}$$

Now, from formula (6.2) and estimates i)-iv) we get

$$|v(t, x)| \leq c(v_o) t^{-\frac{1}{3-2\varepsilon}} |x|^{-\frac{1-2\varepsilon}{3-2\varepsilon}}, (t, x) \in (0, 1) \times \mathbb{R}^3 \setminus B_{R_1}.$$

On the other hand, taking into account (3.5) and that $u = v - w$, then

$$|u(t, x)| \leq c(v_o) t^{-\frac{1}{3-2\varepsilon}} |x|^{-\frac{1-2\varepsilon}{3-2\varepsilon}}, (t, x) \in (0, 1) \times \mathbb{R}^3 \setminus B_{R_1}. \quad (6.6)$$

Employing estimate (4.14) and (6.6), we modify (6.5) as follows

$$\begin{aligned} \|u\|_{L^{\frac{6-4\varepsilon}{1-2\varepsilon}}(|y| > |x|)} & \leq \|u\|_{L^2(6, |x|)}^{\frac{1-2\varepsilon}{3-2\varepsilon}} \|u\|_{L^\infty(|y| > |x|)}^{\frac{2}{3-2\varepsilon}} \\ & \leq c(v_o) t^{\frac{1-2\varepsilon}{6-4\varepsilon} - \frac{2}{(3-2\varepsilon)^2}} |x|^{-\frac{1-2\varepsilon}{6-4\varepsilon} - \frac{2-4\varepsilon}{(3-2\varepsilon)^2}}, t \in (0, 1). \end{aligned} \quad (6.7)$$

We evaluate $\mathbb{T}^{(2)}[u, u](t, \frac{7}{6}x)$ via (6.7):

$$\begin{aligned} |\mathbb{T}^{(2)}[u, u](t, \frac{7}{6}x)| & \leq c \int_0^t \|\nabla T(t, \frac{7}{6}x)\|_{\frac{3}{2}-\varepsilon} \|u\|_{L^{\frac{6-4\varepsilon}{1-2\varepsilon}}(|y| > |x|)}^2 d\tau \\ & \leq c(v_o) |x|^{-\frac{1-2\varepsilon}{3-2\varepsilon} - \frac{4-8\varepsilon}{(3-2\varepsilon)^2}} \int_0^t (t - \tau)^{-\frac{3-4\varepsilon}{3-2\varepsilon}} \tau^{\frac{1-2\varepsilon}{3-2\varepsilon} - \frac{4}{(3-2\varepsilon)^2}} d\tau \\ & \leq c(v_o) t^{-\frac{1+2\varepsilon}{(3-2\varepsilon)^2}} |x|^{-\frac{1-2\varepsilon}{3-2\varepsilon} - \frac{4-8\varepsilon}{(3-2\varepsilon)^2}}, (t, x) \in (0, 1) \times \mathbb{R}^3 \setminus B_{R_1}. \end{aligned} \quad (6.8)$$

Now, from formula (6.2), estimates i)-iii) and estimate (6.8), via (3.5), we modify (6.6) as follows:

$$|u(t, \frac{7}{6}x)| \leq c(v_o)t^{-\frac{1+2\varepsilon}{(3-2\varepsilon)^2}}|x|^{-\frac{1-2\varepsilon}{3-2\varepsilon}-\frac{4-8\varepsilon}{(3-2\varepsilon)^2}}, \quad (t, x) \in (0, 1) \times \mathbb{R}^3 \setminus B_{R_1}. \quad (6.9)$$

Employing estimate (4.14) and (6.9), we modify (6.7) as follows:

$$\begin{aligned} \|u\|_{L^{\frac{6-4\varepsilon}{1-2\varepsilon}}(|y|>\frac{7}{6}|x|)} &\leq \|u\|_{L^2(6,|x|)}^{\frac{1-2\varepsilon}{3-2\varepsilon}} \|u\|_{L^\infty(|y|>\frac{7}{6}|x|)}^{\frac{2}{3-2\varepsilon}} \\ &\leq c(v_o)t^{\frac{1-2\varepsilon}{6-4\varepsilon}-\frac{2+4\varepsilon}{(3-2\varepsilon)^3}}|x|^{-\frac{1-2\varepsilon}{6-4\varepsilon}-\frac{(2-4\varepsilon)}{(3-2\varepsilon)^2}-\frac{8-16\varepsilon}{(3-2\varepsilon)^3}}, \quad t \in (0, 1). \end{aligned} \quad (6.10)$$

By the same arguments we evaluate $\mathbb{T}^{(2)}[u, u](t, \frac{49}{36}x)$ as

$$\begin{aligned} |\mathbb{T}^{(2)}[u, u](t, \frac{49}{36}x)| &\leq c \int_0^t \|T(t-\tau, \frac{7}{6}x)\|_{\frac{3}{2}-\varepsilon} \|u\|_{L^{\frac{6-4\varepsilon}{1-2\varepsilon}}(|y|>\frac{7}{6}|x|)}^2 d\tau \\ &\leq c(v_o)|x|^{-\frac{1-2\varepsilon}{3-2\varepsilon}-\frac{4-8\varepsilon}{(3-2\varepsilon)^2}-\frac{16-32\varepsilon}{(3-2\varepsilon)^3}} \int_0^t (t-\tau)^{-\frac{3-4\varepsilon}{3-2\varepsilon}} \tau^{\frac{1-2\varepsilon}{3-2\varepsilon}-\frac{4+8\varepsilon}{(3-2\varepsilon)^3}} d\tau \\ &\leq c(v_o)t^\sigma |x|^\gamma, \quad (t, x) \in (0, 1) \times \mathbb{R}^3 \setminus B_{R_1}, \end{aligned} \quad (6.11)$$

with $\sigma := \frac{5+4\varepsilon^2-20\varepsilon}{(3-2\varepsilon)^3}$ and $\gamma := -\frac{1-2\varepsilon}{3-2\varepsilon} - \frac{4-8\varepsilon}{(3-2\varepsilon)^2} - \frac{16-32\varepsilon}{(3-2\varepsilon)^3}$. Exponent σ is nonnegative for all $\varepsilon \in (0, \frac{5}{2} - \sqrt{5}]$. We choose $\varepsilon \in (0, \frac{5}{2} - \sqrt{5})$ such that $\gamma = -\frac{3}{2}$. If $\alpha \leq \frac{3}{2}$, then estimate i)-iii) and estimate (6.11) give

$$|v(t, \frac{49}{36}x)| \leq c(v_o)|x|^{-\alpha}, \quad \text{for all } (t, x) \in (0, 1) \times \mathbb{R}^3 \setminus B_{R_1}.$$

Therefore, for $R_2 := \frac{49}{36}R_1$ and $\alpha \in [1, \frac{3}{2}]$,

$$|v(t, x)| \leq c(v_o)|x|^{-\alpha} \quad \text{for all } (t, x) \in (0, 1) \times \mathbb{R}^3 \setminus B_{R_2}. \quad (6.12)$$

If $\alpha > \frac{3}{2}$, then, taking into account estimate (6.12),

$$|v(t, x)| \leq |w(t, x)| + |\mathbb{T}[v, v](t, x)| \leq |w(t, x)| + |\mathbb{T}^{(1)}[v, v](t, x)| + |\mathbb{T}^{(2)}[v, v](t, x)|,$$

$$\mathbb{T}^{(1)}[v, v](t, x) := \int_0^t \int_{|y|<\frac{|x|}{2}} |\nabla T(t-\tau, x-y)| |v(\tau, y)|^2 dy d\tau,$$

$$\mathbb{T}^{(2)}[v, v](t, x) := \int_0^t \int_{|y|>\frac{|x|}{2}} |\nabla T(t-\tau, x-y)| |v(\tau, y)|^2 dy d\tau,$$

$$\text{for all } (t, x) \in (0, 1) \times \mathbb{R}^3 \setminus B_{2R_2}.$$

The term $\mathbb{T}^{(1)}[v, v]$ admits the estimate:

$$|\mathbb{T}^{(1)}[v, v](t, x)| \leq c\|v_o\|_2^2 |x|^{-3}, \quad \text{for all } t \in (0, 1).$$

Since $|x| > 2R_2$, thanks to (6.11) for $\gamma = -\frac{3}{2}$, the term $\mathbb{T}^{(2)}[v, v](t, x)$ admits the estimate:

$$\begin{aligned} |\mathbb{T}^{(2)}[v, v](t, x)| &\leq \int_0^t \|\nabla T(t - \tau, x)\|_1 \|v(\tau)\|_{L^\infty(|y| > \frac{|x|}{2} > R_2)}^2 d\tau \leq c(v_o) |x|^{-3} \int_0^t (t - \tau)^{-\frac{1}{2}} d\tau \\ &\leq c(v_o) |x|^{-3}, \text{ for all } t \in (0, 1]. \end{aligned}$$

Since $\alpha < 3$, the above estimates and (6.12), for $\alpha \in [1, 3)$, prove

$$|v(t, x)| \leq c(v_o) |x|^{-\alpha}, \text{ for all } (t, x) \in (0, 1) \times \mathbb{R}^3 \setminus B_{2R_2}. \quad (6.13)$$

We complete the proof of Theorem 1.2 for $t > 1$.

We consider representation formula (5.6) for $s = 1$. Since the previous arguments ensure that $|v(s, x)| \leq c(v_o) |x|^{-\alpha}$, for all $x \in \mathbb{R}^3 \setminus B_{R_2}$, then, thanks to Lemma 3.1, we easily deduce that

$$|\mathbb{H}[v_i(1)](t - 1, x)| \leq c(v_o) |x|^{-\alpha}, \text{ for all } t > 1 \text{ and } x \in \mathbb{R}^3 \setminus B_{2R_2}. \quad (6.14)$$

So that we evaluate $\mathbb{T}[v, v](t, x)$, whose decomposition is

$$\mathbb{T}[v, v](t, x) := \mathbb{T}^{(1)}[v, v](t, x) + \mathbb{T}^{(2)}[v, v](t, x)$$

with

$$\begin{aligned} \mathbb{T}^{(1)}[v, v](t, x) &:= \int_1^t \int_{|y| < \frac{|x|}{2}} |\nabla T(t - \tau, x - y)| |v(\tau, y)|^2 dy d\tau \\ \mathbb{T}^{(2)}[v, v](t, x) &:= \int_1^t \int_{|y| > \frac{|x|}{2}} |\nabla T(t - \tau, x - y)| |v(\tau, y)|^2 dy d\tau, \text{ for all } (t, x) \in (1, T) \times \mathbb{R}^3 \setminus B_{2R_2}. \end{aligned}$$

We initially consider $\alpha \in [1, 2]$. Taking into account (5.5), the term $\mathbb{T}^{(1)}[v, v]$ admits the estimate:

$$|\mathbb{T}^{(1)}[v, v](t, x)| \leq c \int_1^t (|x|^2 + t - \tau)^{-2} \|v_o\|_2^2 d\tau \leq c(v_o) |x|^{-2}, t > 1$$

If $\alpha \in (2, 3)$, then, in particular we get $v_o \in J^{\frac{3}{2}}(\mathbb{R}^3)$, Hence, by the results of Lemma 2.2,

$$\|v(t)\|_2 \leq c(v_o) t^{-\frac{1}{4}}, t > 0. \quad (6.15)$$

So that, employing (5.5) and (6.15), we are able to estimate $\mathbb{T}^{(1)}[v, v](t, x)$ in the following way:

$$\begin{aligned} |\mathbb{T}^{(1)}[v, v](t, x)| &\leq c \int_1^t \|\nabla T(t - \tau, x)\|_{L^\infty(|y| < \frac{|x|}{2})} \|v(\tau)\|_2^2 d\tau \\ &\leq c |x|^{-3} \int_1^t (t - \tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} d\tau = c(v_o) |x|^{-3}, t > 1. \end{aligned}$$

Therefore, we have proved that

$$\text{for all } \alpha \in [1, 3), \quad |\mathbb{T}^{(1)}[v, v](t, x)| \leq c(v_o)|x|^{-\alpha}, \quad t > 0. \quad (6.16)$$

We consider again the decomposition:

$$\mathbb{T}^{(2)}[v, v](t, x) = \mathbb{T}^{(2)}[u, u](t, x) + \mathbb{T}^{(2)}[u \otimes w](t, x) + \mathbb{T}^{(2)}[w \otimes u](t, x) + \mathbb{T}^{(2)}[w \otimes w](t, x).$$

Moreover, we recall that the estimate of item iii) holds uniformly with respect to t . Hence

$$\begin{aligned} |\mathbb{T}^{(2)}[u \otimes w](t, x)| + |\mathbb{T}^{(2)}[w \otimes u](t, x)| + |\mathbb{T}^{(2)}[w \otimes w](t, x)| &\leq c(v_o)|x|^{-\alpha}, \\ \text{for } \alpha \in [1, 3), \quad t > 1 \text{ and } x \in \mathbb{R}^3 \setminus B_{2R_2}. \end{aligned} \quad (6.17)$$

We consider the $\mathbb{T}^{(2)}[u \otimes u](t, x)$, for which we argue as in item iv). Taking into account that $|x| > 2R_2$, estimate (4.14) and estimate (6.4), employing the interpolation between L^q -spaces, give

$$\|u\|_{L^{\frac{6-4\varepsilon}{1-2\varepsilon}}(|y| > \frac{|x|}{2})} \leq \|u\|_{L^2(2, |x|)}^{\frac{1-2\varepsilon}{3-2\varepsilon}} \|u\|_{L^\infty(|y| > \frac{2}{3}|x|)}^{\frac{2}{3-2\varepsilon}} \leq c(v_o)t^{-\frac{1}{2}\frac{1+2\varepsilon}{3-2\varepsilon}}|x|^{-\frac{1}{2}\frac{1-2\varepsilon}{3-2\varepsilon}}, \quad t > 0. \quad (6.18)$$

By using estimate (5.5) for $T(s, z)$ and estimate (6.18) for u , applying Hölder's inequality, we get

$$\begin{aligned} |\mathbb{T}^{(2)}[u \otimes u](t - \tau, x)| &\leq c \int_1^t \|\nabla T(t - \tau, x)\|_{\frac{3}{2}-\varepsilon} \|u(\tau)\|_{L^{\frac{6-4\varepsilon}{1-2\varepsilon}}(|y| > \frac{|x|}{2})}^2 d\tau \\ &\leq c(v_o)|x|^{-\frac{1-2\varepsilon}{3-2\varepsilon}} \int_1^t (t - \tau)^{-\frac{3-4\varepsilon}{3-2\varepsilon}} \tau^{-\frac{1+2\varepsilon}{3-2\varepsilon}} d\tau \leq c(v_o)t^{-\frac{1}{3-2\varepsilon}}|x|^{-\frac{1-2\varepsilon}{3-2\varepsilon}}, \\ &\quad t > 1 \text{ and } x \in \mathbb{R}^3 \setminus B_{2R_2}. \end{aligned}$$

The last estimate, together with estimates (6.16) and (6.17), furnishes

$$|v(t, x)| \leq c(v_o)|x|^{-\frac{1-2\varepsilon}{3-2\varepsilon}}, \quad t > 1, x \in \mathbb{R}^3 \setminus B_{2R_2}.$$

Now, for the term $\mathbb{T}^{(2)}[v \otimes v]$, we employ a bootstrap argument to realize the exponent α of spatial decay. The last estimate and estimate (2.7) give

$$|v(t, x)|^2 \leq c(v_o)|x|^{-\frac{4}{3}\frac{1-2\varepsilon}{3-2\varepsilon}} \|v(t)\|_\infty^{\frac{2}{3}} \leq c(v_o)|x|^{-\frac{4}{3}\frac{1-2\varepsilon}{3-2\varepsilon}} t^{-\frac{1}{2}}, \quad t > 0, x \in \mathbb{R}^3 \setminus B_{2R_2}. \quad (6.19)$$

Hence, recalling that $\|\nabla T(t - \tau, x)\|_1 \leq c(t - \tau)^{-\frac{1}{2}}$ and (6.1), we get

$$|\mathbb{T}^{(2)}[v \otimes v]| \leq c(v_o)|x|^{-\frac{4}{3}\frac{1-2\varepsilon}{3-2\varepsilon}} \int_1^t \|\nabla T(t - \tau, x)\|_1 \tau^{-\frac{1}{2}} d\tau \leq c(v_o)|x|^{-\frac{4}{3}\frac{1-2\varepsilon}{3-2\varepsilon}}, \quad t > 1, x \in \mathbb{R}^3 \setminus B_{2R_2}.$$

The last estimate, together with estimates (6.16), (6.17), furnishes

$$|v(t, x)| \leq c(v_o)|x|^{-\frac{4}{3}\frac{1-2\varepsilon}{3-2\varepsilon}}, \quad t > 1, x \in \mathbb{R}^3 \setminus B_{2R_2}.$$

Thanks to the last estimate, we modify (6.19) as

$$|v(t, x)|^2 \leq c(v_o)|x|^{-(\frac{4}{3})^2 \frac{1-2\varepsilon}{3-2\varepsilon}} \|v(t)\|_\infty^{\frac{2}{3}} \leq c(v_o)|x|^{-(\frac{4}{3})^2 \frac{1-2\varepsilon}{3-2\varepsilon}} t^{-\frac{1}{2}}, t > 0, x \in \mathbb{R}^3 \setminus B_{2R_2}. \quad (6.20)$$

If we compare (6.19) and (6.20), then, we find that the exponent of spatial decay is increased of a factor $\frac{4}{3}$. Since this can be made in sequence, after a finite number of steps we arrive to an exponent greater or equal than α , proving the final estimate (1.3). The theorem is completely proved. \square

7 Asymptotic time behavior: proof of Corollary 1.1

Thanks to estimate (1.3), to achieve the proof of Corollary 1.1 we can limit ourselves to prove estimate (1.5). Further, it is enough to prove (1.5) for $\beta = \alpha$. The instant $T_0 \leq c\|v_o\|_2^4$, given in Corollary 1.1, is the same of Lemma 2.2 and it is due to Leray in [9]. Moreover, thanks to estimate (2.5), we can prove (1.5) for $\alpha \in (\frac{3}{2}, 3)$. To this end we start from formula (5.7) written for $t > 2T_0$:

$$v_i(t, x) = \mathbb{H}[v_i(T_0)](t - T_0, x) + \int_{T_0}^t (v \cdot \nabla T_i(t, x), v) d\tau, \quad \text{for } (t, x) \in (0, T) \times \mathbb{R}^3. \quad (7.1)$$

From Theorem 1.2 and Lemma 2.2 we get

$$|v(T_0, x)| \leq c(v_o)(1 + |x|)^{-\alpha}, \quad \text{for all } x \in \mathbb{R}^3. \quad (7.2)$$

Then, thanks to the proprieties of the solutions to the Stokes Cauchy problem (see e.g. [7]), we have $|\mathbb{H}[v(T_0)](t - T_0, x)| \leq c(v_o)t^{-\frac{\alpha}{2}}$. Thus, to achieve the result we only need to estimate the nonlinear term in (7.1). We set

$$\int_{T_0}^t (v \cdot \nabla T_i(t, x), v) d\tau = \int_{T_0}^{\frac{t}{2}} (v \cdot \nabla T_i(t, x), v) d\tau + \int_{\frac{t}{2}}^t (v \cdot \nabla T_i(t, x), v) d\tau =: I_1 + I_2.$$

By virtue of (5.5) and (7.2), for all $\alpha \in (\frac{3}{2}, 3)$, we easily deduce the estimate:

$$|I_1| \leq c(v_o) \int_{T_0}^{\frac{t}{2}} \int_{\mathbb{R}^3} (|x - y| + (t - \tau)^{\frac{1}{2}})^{-4} (1 + |y|)^{-2\alpha} dy d\tau \leq c(v_o)t^{-\frac{\alpha}{2}}.$$

Instead, for the term I_2 we achieve the result in two steps. For $\alpha \in (\frac{3}{2}, 2]$, recalling that (2.5) holds, we get

$$\begin{aligned} |I_2| &\leq c(v_o) \int_{\frac{t}{2}}^t \int_{\mathbb{R}^3} (|x - y| + (t - \tau)^{\frac{1}{2}})^{-4} (1 + |y|)^{(-2 + \frac{2}{3}\alpha)\alpha} \tau^{-\frac{\alpha}{2}} dy d\tau \\ &\leq c(v_o)t^{-\frac{\alpha}{2}} \int_{\mathbb{R}^3} |x - y|^{-2} (1 + |y|)^{-\frac{2}{3}\alpha} dy \leq c(v_o)t^{-\frac{\alpha}{2}}. \end{aligned}$$

Hence, by the first step for I_2 and the result for I_1 and for the linear part, we conclude

$$|v(t, x)| \leq c(v_o)t^{-\frac{\alpha}{2}}, \text{ for all } \alpha \in [1, 2].$$

For $\alpha \in (2, 3)$, we invoke this last estimate. Hence, the estimate for I_2 becomes:

$$\begin{aligned} |I_2| &\leq c(v_o) \int_{\frac{t}{2}}^t \int_{\mathbb{R}^3} (|x - y| + (t - \tau)^{\frac{1}{2}})^{-4} (1 + |y|)^{(-2 + \frac{\alpha}{2})\alpha} \tau^{-\frac{\alpha}{2}} dy d\tau \\ &\leq c(v_o) t^{-\frac{\alpha}{2}} \int_{\mathbb{R}^3} |x - y|^{-2} (1 + |y|)^{-\frac{\alpha}{2}} dy \leq c(v_o) t^{-\frac{\alpha}{2}}. \end{aligned}$$

Therefore, the proof is completed.

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